CAR FOLLOWING MODELS

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CHAPTER 4 - Frequently used Symbols

α	=	Numerical coefficients
$a_{\ell,m}$	=	Generalized sensitivity coefficient
$a_f(t)$	=	Instantaneous acceleration of a following vehicle at time t
$a_\ell(t)$	=	Instantaneous acceleration of a lead vehicle at time t
β	=	Numerical coefficient
C	=	Single lane capacity (vehicle/hour)
τ	=	Rescaled time (in units of response time, T)
δ	=	Short finite time period
F	=	Amplitude factor
γ	=	Numerical coefficient
k	=	Traffic stream concentration in vehicles per kilometer
k_j	=	Jam concentration
k_m	=	Concentration at maximum flow
k_f	=	Concentration where vehicle to vehicle interactions begin
k_n	=	Normalized concentration
L	=	Effective vehicle length
L^{-1}	=	Inverse Laplace transform
λ	=	Proportionality factor
λ_i	=	Sensitivity coefficient, i = 1,2,3,...
$ln(x)$	=	Natural logarithm of x
q	=	Flow in vehicles per hour
q_n	=	Normalized flow
$<S>$	=	Average spacing rear bumper to rear bumper
S_i	=	Initial vehicle spacing
S_f	=	Final vehicle spacing
S_o	=	Vehicle spacing for stopped traffic
$S(t)$	=	Inter-vehicle spacing
ΔS	=	Inter-vehicle spacing change
$\overline{T}$	=	Average response time
T_o	=	Propagation time for a disturbance
t	=	Time
t_c	=	Collision time
T	=	Reaction time
U_ℓ	=	Speed of a lead vehicle
U_f	=	Speed of a following vehicle
U_f	=	Final vehicle speed
U_f	=	Free mean speed, speed of traffic near zero concentration
U_i	=	Initial vehicle speed
U_{rel}	=	Relative speed between a lead and following vehicle
$u_\ell(t)$	=	Velocity profile of the lead vehicle of a platoon
V	=	Speed
V_f	=	Final vehicle speed
ω	=	Frequency of a monochromatic speed oscillation
$\ddot{x}_f(t)$	=	Instantaneous acceleration of a following vehicle at time t
$\ddot{x}_\ell(t)$	=	Instantaneous speed of a lead vehicle at time t
$\ddot{x}_f(t)$	=	Instantaneous speed of a following vehicle at time t
$\dot{x}_\ell(t)$	=	Instantaneous speed of a lead vehicle at time t
$\dot{x}_f(t)$	=	Instantaneous speed of a following vehicle at time t
$x_\ell(t)$	=	Instantaneous position of a lead vehicle at time t
$x_f(t)$	=	Instantaneous position of the following vehicle at time t
$x_i(t)$	=	Instantaneous position of the ith vehicle at time t
$z(t)$	=	Position in a moving coordinate system
$< x >$	=	Average of a variable x
Ω	=	Frequency factor

4.
CAR FOLLOWING MODELS

It has been estimated that mankind currently devotes over 10 million man-years each year to driving the automobile, which on demand provides a mobility unequaled by any other mode of transportation. And yet, even with the increased interest in traffic research, we understand relatively little of what is involved in the "driving task". Driving, apart from walking, talking, and eating, is the most widely executed skill in the world today and possibly the most challenging.

Cumming (1963) categorized the various subtasks that are involved in the overall driving task and paralleled the driver's role as an information processor (see Chapter 3). This chapter focuses on one of these subtasks, the task of one vehicle following another on a single lane of roadway (car following). This particular driving subtask is of interest because it is relatively simple compared to other driving tasks, has been successfully described by mathematical models, and is an important facet of driving. Thus, understanding car following contributes significantly to an understanding of traffic flow. Car following is a relatively simple task compared to the totality of tasks required for vehicle control. However, it is a task that is commonly practiced on dual or multiple lane roadways when passing becomes difficult or when traffic is restrained to a single lane. Car following is a task that has been of direct or indirect interest since the early development of the automobile.

One aspect of interest in car following is the average spacing, S, that one vehicle would follow another at a given speed, V. The interest in such speed-spacing relations is related to the fact that nearly all capacity estimates of a single lane of roadway were based on the equation:

$$C = (1000)\, V/S \tag{4.1}$$

where

C = Capacity of a single lane (vehicles/hour)
V = Speed (km/hour)
S = Average spacing rear bumper to rear bumper in meters

The first *Highway Capacity Manual* (1950) lists 23 observational studies performed between 1924 and 1941 that were directed at identifying an operative speed-spacing relation so that capacity estimates could be established for single lanes of roadways. The speed-spacing relations that were obtained from these studies can be represented by the following equation:

$$S = \alpha + \beta V + \gamma V^2 \tag{4.2}$$

where the numerical values for the coefficients, α, β, and γ take on various values. Physical interpretations of these coefficients are given below:

α = the effective vehicle length, L
β = the reaction time, T
γ = the reciprocal of twice the maximum average deceleration of a following vehicle

In this case, the additional term, γV^2, can provide sufficient spacing so that if a lead vehicle comes to a full stop instantaneously, the following vehicle has sufficient spacing to come to a complete stop without collision. A typical value empirically derived for γ would be $\approx$ 0.023 seconds ^{2}/ft . A less conservative interpretation for the non-linear term would be:

$$\gamma = 0.5(a_f^{-1} - a_\ell^{-1}) \tag{4.3}$$

where a_f and a_ℓ are the average maximum decelerations of the following and lead vehicles, respectively. These terms attempt to allow for differences in braking performances between vehicles whether real or perceived (Harris 1964).

For $\gamma = 0$, many of the so-called "good driving" rules that have permeated safety organizations can be formed. In general, the speed-spacing Equation 4.2 attempts to take into account the physical length of vehicles; the human-factor element of perception, decision making, and execution times; and the net physics of braking performances of the vehicles themselves. It has been shown that embedded in these models are theoretical estimates of the speed at maximum flow, $(\alpha/\gamma)^{0.5}$; maximum flow, $[\beta + 2(\alpha\,\gamma)^{0.5}]^{-1}$; and the speed at which small changes in traffic stream speed propagate back through a traffic stream, $(\alpha/\gamma)^{0.5}$ (Rothery 1968).

The speed-spacing models noted above are applicable to cases where each vehicle in the traffic stream maintains the same or nearly the same constant speed and each vehicle is attempting to maintain the same spacing (i.e., it describes a steady-state traffic stream).

Through the work of Reuschel (1950) and Pipes (1953), the dynamical elements of a line of vehicles were introduced. In these works, the focus was on the dynamical behavior of a stream of vehicles as they accelerate or decelerate and each driver-vehicle pair attempts to follow one another. These efforts were extended further through the efforts of Kometani and Sasaki (1958) in Japan and in a series of publications starting in 1958 by Herman and his associates at the General Motors Research Laboratories. These research efforts were microscopic approaches that focused on describing the detailed manner in which one vehicle followed another. With such a description, the macroscopic behavior of single lane traffic flow can be approximated. Hence, car following models form a bridge between individual "car following" behavior and the macroscopic world of a line of vehicles and their corresponding flow and stability properties.

4.1 Model Development

Car following models of single lane traffic assume that there is a correlation between vehicles in a range of inter-vehicle spacing, from zero to about 100 to 125 meters and provides an explicit form for this coupling. The modeling assumes that each driver in a following vehicle is an active and predictable control element in the driver-vehicle-road system. These tasks are termed psychomotor skills or perceptual-motor skills because they require a continued motor response to a continuous series of stimuli.

The relatively simple and common driving task of one vehicle following another on a straight roadway where there is no passing (neglecting all other subsidiary tasks such as steering, routing, etc.) can be categorized in three specific subtasks:

- Perception: The driver collects relevant information mainly through the visual channel. This information arises primarily from the motion of the lead vehicle and the driver's vehicle. Some of the more obvious information elements, only part of which a driver is sensitive to, are vehicle speeds, accelerations and higher derivatives (e.g., "jerk"), inter-vehicle spacing, relative speeds, rate of closure, and functions of these variables (e.g., a "collision time").

- Decision Making: A driver interprets the information obtained by sampling and integrates it over time in order to provide adequate updating of inputs. Interpreting the information is carried out within the framework of a knowledge of vehicle characteristics or class of characteristics and from the driver's vast repertoire of driving experience. The integration of current information and catalogued knowledge allows for the development of driving strategies which become "automatic" and from which evolve "driving skills".

- Control: The skilled driver can execute control commands with dexterity, smoothness, and coordination, constantly relying on feedback from his own responses which are superimposed on the dynamics of the system's counterparts (lead vehicle and roadway).

It is not clear how a driver carries out these functions in detail. The millions of miles that are driven each year attest to the fact that with little or no training, drivers successfully solve a multitude of complex driving tasks. Many of the fundamental questions related to driving tasks lie in the area of 'human factors' and in the study of how human skill is related to information processes.

The process of comparing the inputs of a human operator to that operator's outputs using operational analysis was pioneered by the work of Tustin (1947), Ellson (1949), and Taylor (1949). These attempts to determine mathematical expressions linking input and output have met with limited success. One of the primary difficulties is that the operator (in our case the driver) has no unique transfer function; the driver is a different 'mechanism' under different conditions. While such an approach has met with limited success, through the course of studies like these a number of useful concepts have been developed. For example, reaction times were looked upon as characteristics of individuals rather than functional characteristics of the task itself. In addition, by introducing the concept of "information", it has proved possible to parallel reaction time with the rate of coping with information.

The early work by Tustin (1947) indicated maximum rates of the order of 22-24 bits/second (sec). Knowledge of human performance and the rates of handling information made it possible to design the response characteristics of the machine for maximum compatibility of what really is an operator-machine system.

The very concept of treating an operator as a transfer function implies, partly, that the operator acts in some continuous manner. There is some evidence that this is not completely correct and that an operator acts in a discontinuous way. There is a period of time during which the operator having made a "decision" to react is in an irreversible state and that the response must follow at an appropriate time, which later is consistent with the task.

The concept of a human behavior being discontinuous in carrying out tasks was first put forward by Uttley (1944) and has been strengthened by such studies as Telfor's (1931), who demonstrated that sequential responses are correlated in such a way that the response-time to a second stimulus is affected significantly by the separation of the two stimuli. Inertia, on the other hand, both in the operator and the machine, creates an appearance of smoothness and continuity to the control element.

In car following, inertia also provides direct feedback data to the operator which is proportional to the acceleration of the vehicle. Inertia also has a smoothing effect on the performance requirements of the operator since the large masses and limited output of drive-trains eliminate high frequency components of the task.

Car following models have not explicitly attempted to take all of these factors into account. The approach that is used assumes that a stimulus-response relationship exists that describes, at least phenomenologically, the control process of a driver-vehicle unit. The stimulus-response equation expresses the concept that a driver of a vehicle responds to a given stimulus according to a relation:

$$\text{Response} = \lambda \text{ Stimulus} \tag{4.4}$$

where λ is a proportionality factor which equates the stimulus function to the response or control function. The stimulus function is composed of many factors: speed, relative speed, inter-vehicle spacing, accelerations, vehicle performance, driver thresholds, etc.

Do all of these factors come into play part of the time? The question is, which of these factors are the most significant from an explanatory viewpoint. Can any of them be neglected and still retain an approximate description of the situation being modeled?

What is generally assumed in car following modeling is that a driver attempts to: (a) keep up with the vehicle ahead and (b) avoid collisions.

These two elements can be accomplished if the driver maintains a small average relative speed, U_{rel} over short time periods, say δt, i.e.,

$$\langle U_{\ell} - U_f \rangle = \langle U_{rel} \rangle \frac{1}{\delta t}\int_{t-\delta t/2}^{t+\delta t/2} U_{rel}(t)\,dt \tag{4.5}$$

is kept small. This ensures that 'collision' times:

$$t_c = \frac{S(t)}{U_{rel}} \tag{4.6}$$

are kept large, and inter-vehicle spacings would not appreciably increase during the time period, δt. The duration of the δt will depend in part on alertness, ability to estimate quantities such as: spacing, relative speed, and the level of information required for the driver to assess the situation to a tolerable probability level (e.g., the probability of detecting the relative movement of an object, in this case a lead vehicle) and can be expressed as a function of the perception time.

Because of the role relative-speed plays in maintaining relatively large collision times and in preventing a lead vehicle from 'drifting' away, it is assumed as a first approximation that the argument of the stimulus function is the relative speed.

From the discussion above of driver characteristics, relative speed should be integrated over time to reflect the recent time history of events, i.e., the stimulus function should have the form like that of Equation 4.5 and be generalized so that the stimulus at a given time, t, depends on the weighted sum of all earlier values of the relative speed, i.e.,

$$< U_\ell - U_f > \; = \; < U_{rel} > = \int_{t-\delta t/2}^{t+\delta t/2} \sigma(t-t')U_{rel}(t')dt \qquad (4.7)$$

where $\sigma(t)$ is a weighing function which reflects a driver's estimation, evaluation, and processing of earlier information (Chandler et al. 1958). The driver weighs past and present information and responds at some future time. The consequence of using a number of specific weighing functions has been examined (Lee 1966), and a spectral analysis approach has been used to derive a weighing function directly from car following data (Darroch and Rothery 1969).

The general features of a weighting function are depicted in Figure 4.1. What has happened a number of seconds ($\approx$ 5 sec) in the past is not highly relevant to a driver now, and for a short time ($\approx$ 0.5 sec) a driver cannot readily evaluate the information available to him. One approach is to assume that

$$\sigma(t) = \delta(t-T) \qquad (4.8)$$

where

$$\delta(t-T) = 0, \; for \;\; t \neq T \qquad (4.9)$$

$$\delta(t-T) = 1, \; for \;\; t = T \qquad (4.10)$$

and

$$\int_o^\infty \delta(t-T)dt = 1$$

For this case, our stimulus function becomes

$$\text{Stimulus}(t) = U_\ell(t - T) - U_f(t - T) \qquad (4.11)$$

which corresponds to a simple constant response time, T, for a driver-vehicle unit. In the general case of $\sigma(t)$, there is an average response time, $\overline{T}$, given by

$$\overline{T(t)} = \int_0^t t'\sigma(t')dt' \qquad (4.12)$$

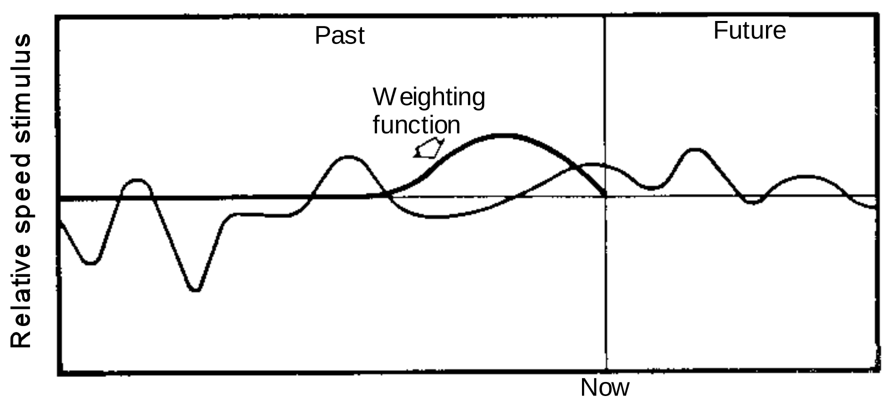


Figure 4.1
Schematic Diagram of Relative Speed Stimulus and a Weighting Function Versus Time (Darroch and Rothery 1972).

The main effect of such a response time or delay is that the driver is responding at all times to a stimulus. The driver is observing the stimulus and determining a response that will be made some time in the future. By delaying the response, the driver obtains "advanced" information.

For redundant stimuli there is little need to delay response, apart from the physical execution of the response. Redundancy alone can provide advance information and for such cases, response times are shorter.

The response function is taken as the acceleration of the following vehicle, because the driver has direct control of this quantity through the 'accelerator' and brake pedals and also because a driver obtains direct feedback of this variable through inertial forces, i.e.,

$$\text{Response } (t) = a_f(t) = \ddot{x}_f(t) \tag{4.13}$$

where $x_i(t)$ denotes the longitudinal position along the roadway of the ith vehicle at time t. Combining Equations4.11 and 4.13 into Equation 4.4 the stimulus-response equation becomes (Chandler et al. 1958):

$$\ddot{x}_f(t) = \lambda[\dot{x}_\ell(t-T) - \dot{x}_f(t-T)] \tag{4.14}$$

or equivalently

$$\ddot{x}_f(t+T) = \lambda[\dot{x}_\ell(t) - \dot{x}_f(t)] \tag{4.15}$$

Equation 4.15 is a first approximation to the stimulus-response equation of car-following, and as such it is a grossly simplified description of a complex phenomenon. A generalization of car following in a conventional control theory block diagram is shown in Figure 4.1a. In this same format the linear car-following model presented in Equation 4.15 is shown in Figure 4.1b. In this figure the driver is represented by a time delay and a gain factor. Undoubtedly, a more complete representation of car following includes a set of equations that would model the dynamical properties of the vehicle and the roadway characteristics. It would also include the psychological and physiological properties of drivers, as well as couplings between vehicles, other than the forward nearest neighbors and other driving tasks such as lateral control, the state of traffic, and emergency conditions.

For example, vehicle performance undoubtedly alters driver behavior and plays an important role in real traffic where mixed traffic represents a wide performance distribution, and where appropriate responses cannot always be physically achieved by a subset of vehicles comprising the traffic stream. This is one area where research would contribute substantially to a better understanding of the growth, decay, and frequency of disturbances in traffic streams (see, e.g., Harris 1964; Herman and Rothery 1967; Lam and Rothery 1970).

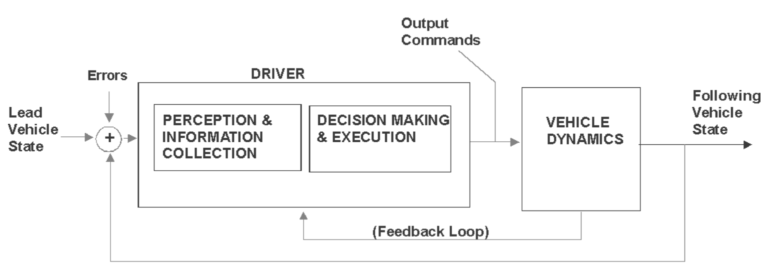


Figure 4.1a
Block Diagram of Car-Following.

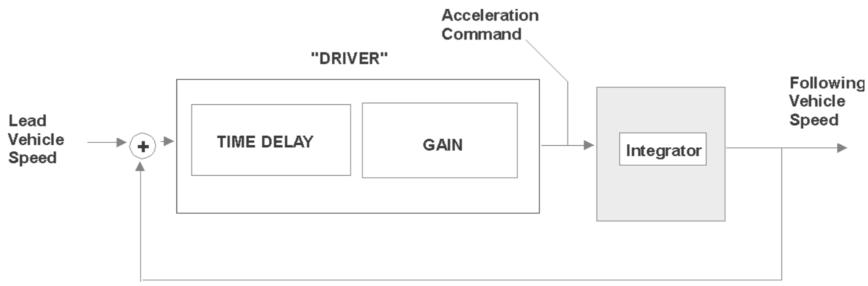


Figure 4.1b
Block Diagram of the Linear Car-Following Model.

4.2 Stability Analysis

In this section we address the stability of the linear car following equation, Equation 4.15, with respect to disturbances. Two particular types of stabilities are examined: local stability and asymptotic stability.

Local Stability is concerned with the response of a following vehicle to a fluctuation in the motion of the vehicle directly in front of it; i.e., it is concerned with the localized behavior between pairs of vehicles.

Asymptotic Stability is concerned with the manner in which a fluctuation in the motion of any vehicle, say the lead vehicle of a platoon, is propagated through a line of vehicles.

The analysis develops criteria which characterize the types of possible motion allowed by the model. For a given range of model parameters, the analysis determines if the traffic stream (as described by the model) is stable or not, (i.e., whether disturbances are damped, bounded, or unbounded). This is an important determination with respect to understanding the applicability of the modeling. It identifies several characteristics with respect to single lane traffic flow, safety, and model validity. If the model is realistic, this range should be consistent with measured values of these parameters in any applicable situation where disturbances are known to be stable. It should also be consistent with the fact that following a vehicle is an extremely common experience, and is generally stable.

4.2.1 Local Stability

In this analysis, the linear car following equation, (Equation 4.15) is assumed. As before, the position of the lead vehicle and the following vehicle at a time, t, are denoted by $x_\ell(t)$ and $x_f(t)$, respectively. Rescaling time in units of the response time, T, using the transformation, $t = \tau T$, Equation 4.15 simplifies to

$$\ddot{x}_f(\tau+1) = C[(\dot{x}_\ell(\tau) - \dot{x}_f(\tau))] \qquad (4.16)$$

where $C = \lambda T$. The conditions for the local behavior of the following vehicle can be derived by solving Equation 4.16 by the method of Laplace transforms (Herman et al. 1959).

The evaluation of the inverse Laplace transform for Equation 4.16 has been performed (Chow 1958; Kometani and Sasaki 1958). For example, for the case where the lead and following vehicles are initially moving with a constant speed, u, the solution for the speed of the following vehicle was given by Chow where ν denotes the integral part of t/T. The complex form of Chow's solution makes it difficult to describe various physical properties (Chow 1958).

$$\dot{x}_n(t) = u + \sum_{\sigma=0}^{\nu-n} \int_{(n+\sigma)T}^{t} (-1)^{\sigma} \lambda^{n+\sigma} \frac{\left|\left|\tau-(n+\sigma)T\right|\right|^{n+\sigma-1}}{(n-1)!\sigma!} \bullet (u_0(t-\tau)-u)\,dt$$

However, the general behavior of the following vehicle's motion can be characterized by considering a specific set of initial conditions. Without any loss in generality, initial conditions are assumed so that both vehicles are moving with a constant speed, u. Then using a moving coordinate system $z(t)$ for both the lead and following vehicles the formal solution for the acceleration of the following vehicle is given more simply by:

$$L^{-1}[C(C + se^{s})^{-1}s \tag{4.16a}$$

where L^{-1} denotes the inverse Laplace transform. The character of the above inverse Laplace transform is determined by the singularities of the factor $(C + se^s)^{-1}$ since $Cs^2Z_\ell(s)$ is a regular function. These singularities in the finite plane are the simple poles of the roots of the equation

$$C + se^{s} = 0 \tag{4.17}$$

Similarly, solutions for vehicle speed and inter-vehicle spacings can be obtained. Again, the behavior of the inter-vehicle spacing is dictated by the roots of Equation 4.17. Even for small t, the character of the solution depends on the pole with the largest real part, say, $s_0 = a_0 + ib_0$, since all other poles have considerably larger negative real parts so that their contributions are heavily damped.

Hence, the character of the inverse Laplace transform has the form $e^{a_0 t}\, e^{tb_0 t}$. For different values of C, the pole with the largest real part generates four distinct cases:

a) if $C \le e^{-1} (\approx 0.368)$, then $a_0 \le 0$, $b_0 = 0$, and the motion is non-oscillatory and exponentially damped.

b) if $e^{-1} < C < \pi/2$, then $a_0 < 0$, $b_0 > 0$ and the motion is oscillatory with exponential damping.

c) if $C = \pi/2$, then $a_0 = 0$, b_0, > 0 and the motion is oscillatory with constant amplitude.

d) if $C > \pi/2$ then $a_0 > 0$, b_0 > 0 and the motion is oscillatory with increasing amplitude.

The above establishes criteria for the numerical values of C which characterize the motion of the following vehicle. In particular, it demonstrates that in order for the following vehicle not to over-compensate to a fluctuation, it is necessary that $C \le 1/e$. For values of C that are somewhat greater, oscillations occur but are heavily damped and thus insignificant. Damping occurs to some extent as long as $C < \pi/2$.

These results concerning the oscillatory and non-oscillatory behavior apply to the speed and acceleration of the following vehicle as well as to the inter-vehicle spacing. Thus, e.g., if $C \le e^{-1}$, the inter-vehicle spacing changes in a non-oscillatory manner by the amount ΔS, where

$$\Delta S = \frac{1}{\lambda}(V - U) \tag{4.18}$$

when the speeds of the vehicle pair changes from U to V. An important case is when the lead vehicle stops. Then, the final speed, V, is zero, and the total *change* in inter-vehicle spacing is $-U/\lambda$.

In order for a following vehicle to avoid a 'collision' from initiation of a fluctuation in a lead vehicle's speed the inter-vehicle spacing should be at least as large as U/λ. On the other hand, in the interests of traffic flow the inter-vehicle spacing should be small by having λ as large as possible and yet within the stable limit. Ideally, the best choice of λ is $(eT)^{-1}$.

The result expressed in Equation 4.18 follows directly from Chow's solution (or more simply by elementary considerations). Because the initial and final speeds for both vehicles are U and V, respectively, we have

$$\int_0^\infty \ddot{x}_f(t+T)dt = V - U \tag{4.19}$$

and from Equation 4.15 we have

$$\lambda \int_0^\infty [\dot{x}_l(t) - \dot{x}_f(t)]dt = \Delta S$$

or

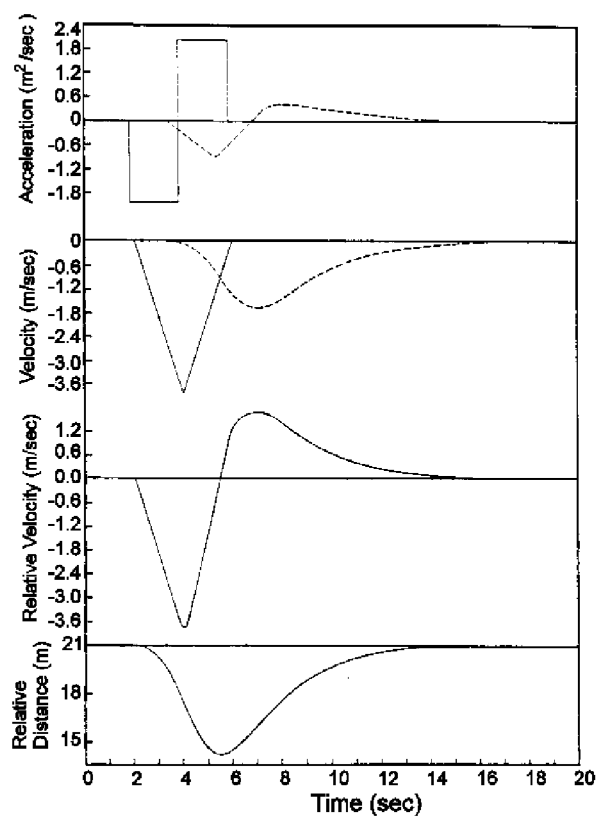
$$\Delta S = \int_0^\infty [\dot{x}_\ell(t) - \dot{x}_f(t)]dt = \frac{V-U}{\lambda} \tag{4.20}$$

as given earlier in Equation 4.18.

In order to illustrate the general theory of local stability, the results of several calculations using a Berkeley Ease analog computer and an IBM digital computer are described. It is interesting to note that in solving the linear car following equation for two vehicles, estimates for the local stability condition were first obtained using an analog computer for different values of C which differentiate the various type of motion.

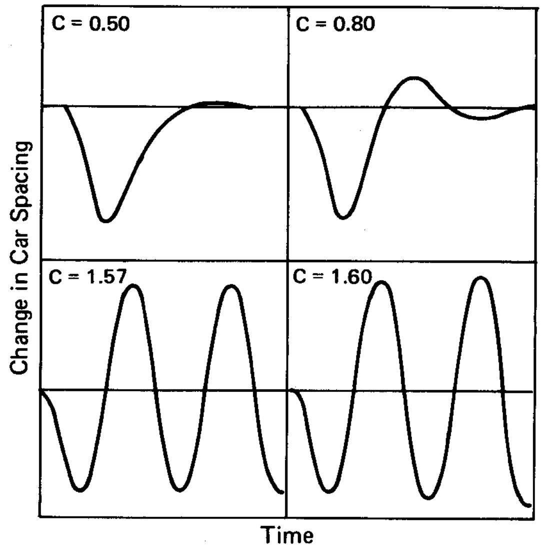
Figure 4.2 illustrates the solutions for $C= e^{-1}$, where the lead vehicle reduces its speed and then accelerates back to its original speed. Since C has a value for the locally stable limit, the acceleration and speed of the following vehicle, as well as the inter-vehicle spacing between the two vehicles are non-oscillatory.

In Figure 4.3, the inter-vehicle spacing is shown for four other values of C for the same fluctuation of the lead vehicle as shown in Figure 4.2. The values of C range over the cases of oscillatory



*Note: Vehicle 2 follows Vehicle 1 (lead car) with a time lag T=1.5 sec and a value of $C=e^{-1}$(≈0.368), the limiting value for local stability. The initial velocity of each vehicle is **u***

Figure 4.2
Detailed Motion of Two Cars Showing the
Effect of a Fluctuation in the Acceleration of the Lead Car (Herman et al. 1958).



Note: Changes in car spacings from an original constant spacing between two cars for the noted values of C. The acceleration profile of the lead car is the same as that shown in Figure 4.2.

Figure 4.3
Changes in Car Spacings from an Original Constant Spacing Between Two Cars (Herman et al. 1958).

motion where the amplitude is damped, undamped, and of increasing amplitude.

For the values of C = 0.5 and 0.80, the spacing is oscillatory and heavily damped.

$$\text{For } C = 1.57 \ (\approx \frac{\pi}{2}),$$

the spacing oscillates with constant amplitude. For C = 1.60, the motion is oscillatory with increasing amplitude.

Local Stability with Other Controls. Qualitative arguments can be given of a driver's lack of sensitivity to variation in relative acceleration or higher derivatives of inter-vehicle spacings because of the inability to make estimates of such quantities. It is of interest to determine whether a control centered around such derivatives would be locally stable. Consider the car following equation of the form

$$\ddot{x}_f(\tau+1) = C\frac{d^m}{dt^m}[x_\ell(\tau)-x_f(\tau)] \tag{4.21}$$

for m= 0,1,2,3..., i.e., a control where the acceleration of the following vehicle is proportional to the mth derivative of the inter-vehicle spacing. For m = 1, we obtain the linear car following equation.

Using the identical analysis for any m, the equation whose roots determine the character of the motion which results from Equation 4.21 is

$$C + s^m e^s = 0 \tag{4.22}$$

None of these roots lie on the negative real axis when m is even, therefore, local stability is possible only for odd values of the mth derivative of spacing: relative speed, the first derivative of relative acceleration (m = 3), etc. Note that this result indicates that an acceleration response directly proportional to inter-vehicle spacing stimulus is unstable.

4.2.2 Asymptotic Stability

In the previous analysis, the behavior of one vehicle following another was considered. Here a platoon of vehicles (except for the platoon leader) follows the vehicle ahead according to the linear car following equation, Equation 4.15. The criteria necessary for asymptotic stability or instability were first investigated by considering the Fourier components of the speed fluctuation of a platoon leader (Chandler et al. 1958).

The set of equations which attempts to describe a line of N identical car-driver units is:

$$\ddot{x}_{n+1}(t+T) = \lambda[\dot{x}_n(t) - \dot{x}_{n+1}(t)] \tag{4.23}$$

where $n = 0,1,2,3,...N$.

Any specific solution to these equations depends on the velocity profile of the lead vehicle of the platoon, $u_0(t)$, and the two parameters λ and T. For any inter-vehicle spacing, if a disturbance grows in amplitude then a 'collision' would eventually occur somewhere back in the line of vehicles.

While numerical solutions to Equation 4.23 can determine at what point such an event would occur, the interest is to determine criteria for the growth or decay of such a disturbance. Since an arbitrary speed pattern can be expressed as a linear combination of monochromatic components by Fourier analysis, the specific profile of a platoon leader can be simply represented by one component, i.e., by a constant together with a monochromatic oscillation with frequency, ω and amplitude, f_o, i.e.,

$$u_o(t) = a_o + f_o e^{i\omega t} \tag{4.24}$$

and the speed profile of the nth vehicle by

$$u_n(t) = a_o + f_n e^{i\omega t} \tag{4.25}$$

Substitution of Equations 4.24 and 4.25 into Equation 4.23 yields:

$$u_n(t) = a_o + F(\omega,\lambda,T,n)e^{i\Omega(\omega,\lambda,T,n)} \tag{4.26}$$

where the amplitude factor $F(\omega, \lambda, T, n)$ is given by

$$[1 + (\frac{\omega}{\lambda})^2 + 2(\frac{\omega}{\lambda})\sin(\omega T)]^{-n/2}$$

which decreases with increasing n if

$$1 + (\frac{\omega}{\lambda})^2 + 2(\frac{\omega}{\lambda})\sin(\omega T) > 1$$

i.e. if

$$\frac{\omega}{\lambda} > 2\sin(\omega T)$$

The severest restriction on the parameter λ arises from the low frequency range, since in the limit as $\omega \to 0$, λ must satisfy the inequality

$$\lambda T < \frac{1}{2}[\lim_{\omega \to 0}(\omega T)/\sin(\omega T)] \tag{4.27}$$

Accordingly, asymptotic stability is insured for all frequencies where this inequality is satisfied.

For those values of ω within the physically realizable frequency range of vehicular speed oscillations, the right hand side of the inequality of 4.27 has a short range of values of 0.50 to about 0.52. The asymptotic stability criteria divides the two parameter domain into stable and unstable regions, as graphically illustrated in Figure 4.4.

The criteria for local stability (namely that no local oscillations occur when $\lambda T \leq e^{-1}$) also insures asymptoticstability. It has also been shown (Chandler et al. 1958) that a speed fluctuation can be approximated by:

$$x_{n+1}(t) \approx u_0(t)\left[\frac{4\pi n}{\lambda}\left(\frac{1}{2\lambda} - T\right)\right]^{-1/2} \cdot \exp\left[\frac{[t - n/\lambda]}{4n/\lambda(1/2\lambda - T)}\right] \tag{4.28}$$

Hence, the speed of propagation of the disturbance with respect to the moving traffic stream in number of inter-vehicle separations per second, n/t, is λ.

That is, the time required for the disturbance to propagate between pairs of vehicles is λ^{-1}, a constant, which is independent of the response time T. It is noted from the above equation that in the propagation of a speed fluctuation the amplitude of the

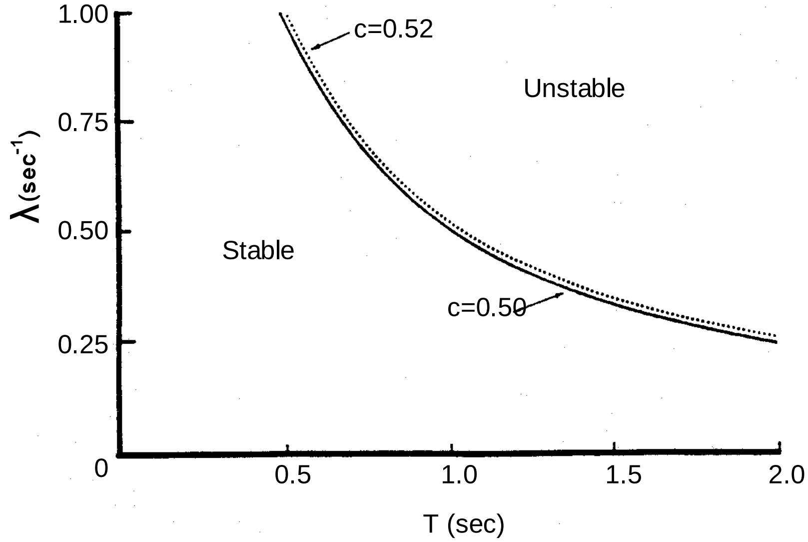


Figure 4.4
Regions of Asymptotic Stability (Rothery 1968).

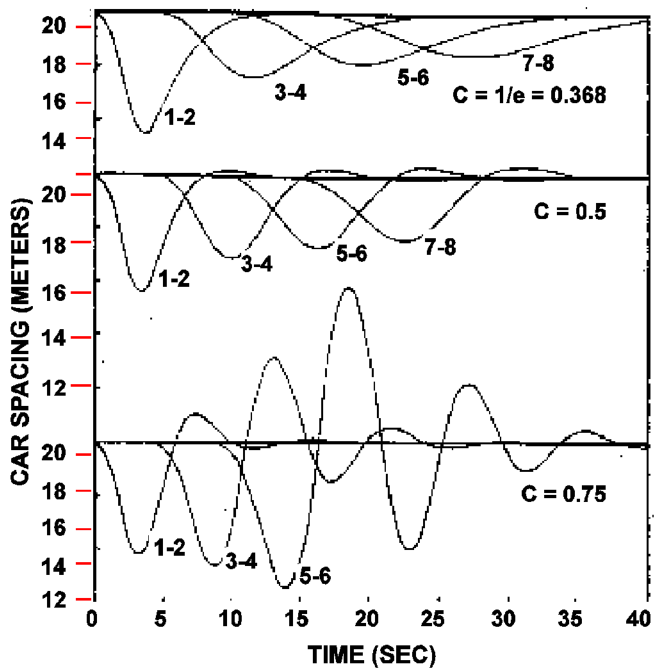
disturbance grows as the response time, T, approaches $1/(2\lambda)$ until instability is reached. Thus, while $\lambda T < 0.5$ ensures stability, short reaction times increase the range of the sensitivity coefficient, λ, that ensures stability. From a practical viewpoint, small reaction times also reduce relatively large responses to a given stimulus, or in contrast, larger response times require relatively large responses to a given stimulus. Acceleration fluctuations can be correspondingly analyzed (Chandler et al. 1958).

4.2.1.1 Numerical Examples

In order to illustrate the general theory of asymptotic stability as outlined above, the results of a number of numerical calculations are given. Figure 4.5 graphically exhibits the inter-vehicle spacings of successive pairs of vehicles versus time for a platoon of vehicles. Here, three values of C were used: $C = 0.368$, 0.5, and 0.75. The initial fluctuation of the lead vehicle, $n = 1$, was the same as that of the lead vehicle illustrated in Figure 4.2. This disturbance consists of a slowing down and then a speeding up to the original speed so that the integral of acceleration over time is zero. The particularly stable, non-oscillatory response is evident in the first case where $C = 0.368$ ($\approx 1/e$), the local stability limit. As analyzed, a heavily damped oscillation occurs in the second case where $C = 0.5$, the asymptotic limit. Note that the amplitude of the disturbance is damped as it propagates through the line of vehicles even though this case is at the asymptotic limit.

This results from the fact that the disturbance is not a single Fourier component with near zero frequency. However, instability is clearly exhibited in the third case of Figure 4.5 where $C = 0.75$ and in Figure 4.6 where $C = 0.8$. In the case shown in Figure 4.6, the trajectories of each vehicle in a platoon of nine are graphed with respect to a coordinate system moving with the initial platoon speed u. Asymptotic instability of a platoon of nine cars is illustrated for the linear car following equation, Equation 4.23, where $C = 0.80$. For $t = 0$, the vehicles are all moving with a velocity u and are separated by a distance of 12 meters. The propagation of the disturbance, which can be readily discerned, leads to "collision" between the 7th and 8th cars at about $t = 24$ sec. The lead vehicle at $t = 0$ decelerates for 2 seconds at 4 km/h/sec, so that its speed changes from u to u -8 km/h and then accelerates back to u. This fluctuation in the speed of the lead vehicle propagates through the platoon in an unstable manner with the inter-vehicle spacing between the seventh and eighth vehicles being reduced to zero at about 24.0 sec after the initial phase of the disturbance is generated by the lead vehicle of the platoon.

In Figure 4.7 the envelope of the minimum spacing that occurs between successive pairs of vehicles is graphed versus time



Note: Diagram uses Equation 4.23 for three values of C. The fluctuation in acceleration of the lead car, car number 1, is the same as that shown in Fig. 4.2 At t=0 the cars are separated by a spacing of 21 meters.

Figure 4.5
Inter-Vehicle Spacings of a Platoon of Vehicles
Versus Time for the Linear Car Following Model (Herman et al. 1958).

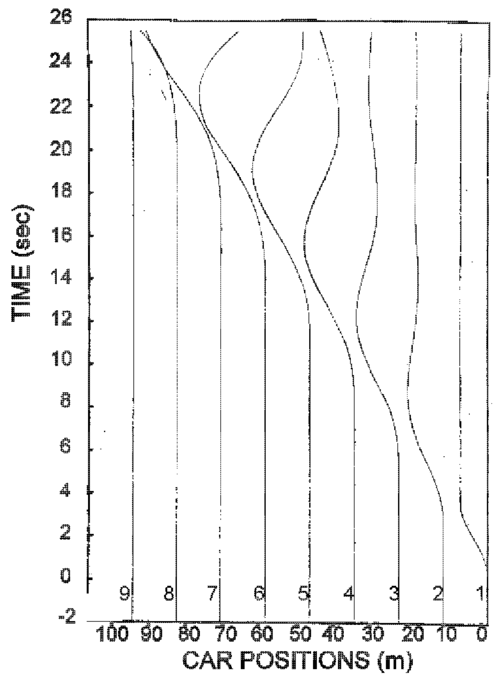
where the lead vehicle's speed varies sinusoidally with a frequency $\omega = 2\pi/10$ radian/sec. The envelope of minimum inter-vehicle spacing versus vehicle position is shown for three values of λ. The response time, T, equals 1 second. It has been shown that the frequency spectrum of relative speed and acceleration in car following experiments have essentially all their content below this frequency (Darroch and Rothery 1973).

The values for the parameter λ were 0.530, 0.5345, and 0.550/sec. The value for the time lag, T, was 1 sec in each case. The frequency used is that value of ω which just satisfies the stability equation, Equation 4.27, for the case where λ= 0.5345/sec. This latter figure serves to demonstrate not only the stability criteria as a function of frequency but the accuracy of the numerical results. A comparison between that which is predicted from the stability analysis and the numerical solution for the constant amplitude case (λ=0.5345/sec) serves as a check point. Here, the numerical solution yields a maximum and minimum amplitude that is constant to seven significant places.

4.2.1.2 Next-Nearest Vehicle Coupling

In the nearest neighbor vehicle following model, the motion of each vehicle in a platoon is determined solely by the motion of the vehicle directly in front. The effect of including the motion of the "next nearest neighbor" vehicle (i.e., the car which is two vehicles ahead in addition to the vehicle directly in front) can be ascertained. An approximation to this type of control, is the model

$$\ddot{x}_{n+2}(t+\mathrm{T}) = \lambda_1[\dot{x}_{n+1}(t) - \dot{x}_{n+2(t)}] + \lambda_2[\dot{x}_n(t) - \dot{x}_{n+2}] \qquad (4.29)$$



Note: Diagram illustrates the linear car following equation, eq. 4.23, where C=080.

Figure 4.6
Asymptotic Instability of a Platoon of Nine Cars (Herman et al. 1958).

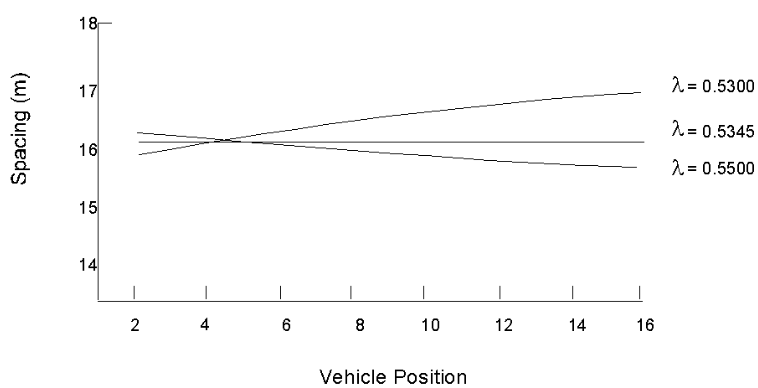


Figure 4.7
Envelope of Minimum Inter-Vehicle Spacing Versus Vehicle Position (Rothery 1968).

$$(\lambda_1 + \lambda_2)T < \frac{1}{2}(\omega T)/\sin(\omega T)] \quad (4.30)$$

which in the limit $\omega \rightarrow 0$ is

$$(\lambda_1 + \lambda_2)T > \frac{1}{2} \quad (4.31)$$

This equation states that the effect of adding next nearest neighbor coupling to the control element is, to the first order, to increase λ_1 to $(\lambda_1 + \lambda_2)$. This reduces the value that λ_1 can have and still maintain asymptotic stability.

4.3 Steady-State Flow

This section discusses the properties of steady-state traffic flow based on car following models of single-lane traffic flow. In particular, the associated speed-spacing or equivalent speed-concentration relationships, as well as the flow-concentration relationships for single lane traffic flow are developed.

The Linear Case. The equations of motion for a single lane of traffic described by the linear car following model are given by:

$$\ddot{x}_{n+1}(t+T) = \lambda[\dot{x}_n(t) - \dot{x}_{n+1}(t)] \quad (4.32)$$

where n = 1, 2, 3, ...

In order to interrelate one steady-state to another under this control, assume (up to a time t=0) each vehicle is traveling at a speed U_i and that the inter-vehicle spacing is S_i. Suppose that at t=0, the lead vehicle undergoes a speed change and increases or decreases its speed so that its final speed after some time, t, is U_f. A specific numerical solution of this type of transition is exhibited in Figure 4.8.

In this example $C = \lambda T$=0.47 so that the stream of traffic is stable, and speed fluctuations are damped. Any case where the asymptotic stability criteria is satisfied assures that each following vehicle comprising the traffic stream eventually reaches a state traveling at the speed U_f. In the transition from a speed U_i to a speed U_f, the inter-vehicle spacing S changes from S_i to S_f, where

$$S_f = S_i + \lambda^{-1}(U_f - U_i) \quad (4.33)$$

This result follows directly from the solution to the car following equation, Equation 4.16a or from Chow (1958). Equation 4.33 also follows from elementary considerations by integration of Equation 4.32 as shown in the previous section (Gazis et al. 1959). This result is not directly dependent on the time lag, T, except that for this result to be valid the time lag, T, must allow the equation of motion to form a stable stream of traffic. Since vehicle spacing is the inverse of traffic stream concentration, k, the speed-concentration relation corresponding to Equation 4.33 is:

$$k_f^{-1} = k_i^{-1} + \lambda^{-1}(U_f - U_i) \quad (4.34)$$

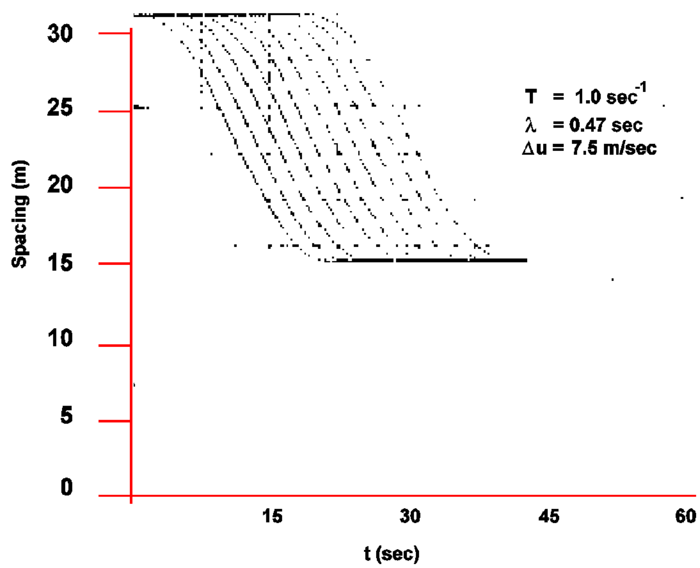
The significance of Equations 4.33 and 4.34 is that:

1) They link an initial steady-state to a second arbitrary steady-state, and

2) They establish relationships between macroscopic traffic stream variables involving a microscopic car following parameter, λ .

In this respect they can be used to test the applicability of the car following model in describing the overall properties of single lane traffic flow. For stopped traffic, U_i = 0, and the corresponding spacing, S_o, is composed of vehicle length and "bumper-to-bumper" inter-vehicle spacing. The concentration corresponding to a spacing, S_o, is denoted by k_j and is frequently referred to as the 'jam concentration'.

Given k_j, then Equation 4.34 for an arbitrary traffic state defined by a speed, U, and a concentration, k, becomes



Note: A numerical solution to Equation 4.32 for the inter-vehicle spacings of an 11- vehicle platoon going from one steady-state to another ($\lambda T = 0.47$). The lead vehicle's speed decreases by 7.5 meters per second.

Figure 4.8
Inter-Vehicle Spacings of an Eleven Vehicle Platoon (Rothery 1968).

$$U = \lambda(k^{-1} - k_j^{-1}) \tag{4.35}$$

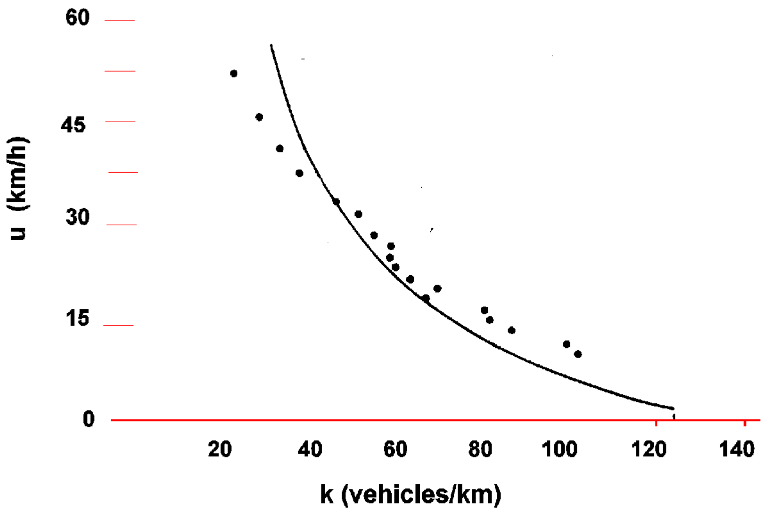
A comparison of this relationship was made (Gazis et al. 1959) with a specific set of reported observations (Greenberg 1959) for a case of single lane traffic flow (i.e., for the northbound traffic flowing through the Lincoln Tunnel which passes under the Hudson River between the States of New York and New Jersey). This comparison is reproduced in Figure 4.9 and leads to an estimate of 0.60 sec^{-1} for λ. This estimate of λ implies an upper bound for T $\approx$ 0.83 sec for an asymptotic stable traffic stream using this facility.

While this fit and these values are not unreasonable, a fundamental problem is identified with this form of an equation for a speed-spacing relationship (Gazis et al. 1959). Because it is linear, this relationship does not lead to a reasonable description of traffic flow. This is illustrated in Figure 4.10 where the same data from the Lincoln Tunnel (in Figure 4.9) is regraphed. Here the graph is in the form of a normalized flow, versus a normalized concentration together with the corresponding theoretical steady-state result derived from Equation 4.35, i.e.,

$$q = Uk = \lambda\left(1 - \frac{k}{k_j}\right) \tag{4.36}$$

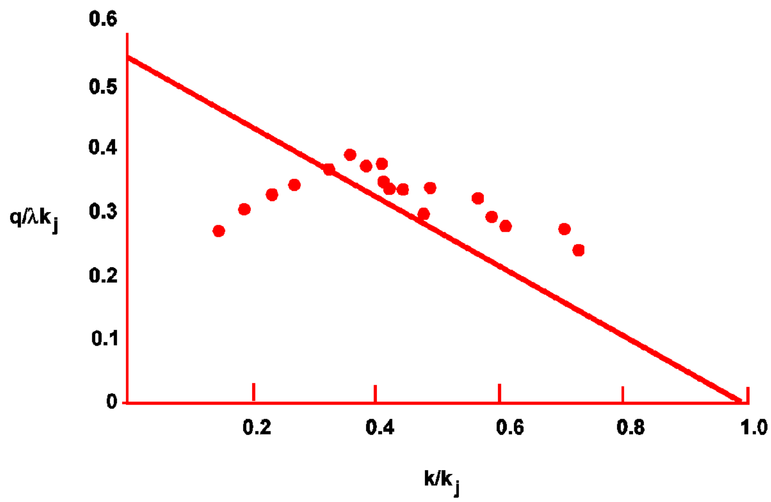
The inability of Equation 4.36 to exhibit the required qualitative relationship between flow and concentration (see Chapter 2) led to the modification of the linear car following equation (Gazis et al. 1959).

Non-Linear Models. The linear car following model specifies an acceleration response which is completely independent of vehicle spacing (i.e., for a given relative velocity, response is the same whether the vehicle following distance is small [e.g., of the order of 5 or 10 meters] or if the spacing is relatively large [i.e., of the order of hundreds of meters]). Qualitatively, we would expect that response to a given relative speed to increase with smaller spacings.



Note: The data are those of (Greenberg 1959) for the Lincoln Tunnel. The curve represents a "least squares fit" of Equation 4.35 to the data.

Figure 4.9
Speed (miles/hour) Versus Vehicle Concentration (vehicles/mile).(Gazis et al. 1959).



Note: The curve corresponds to Equation 4.36 where the parameters are those from the "fit" shown in Figure 4.9.

Figure 4.10
Normalized Flow Versus Normalized Concentration (Gazis et al. 1959).

In order to attempt to take this effect into account, the linear model is modified by supposing that the gain factor, λ, is not a constant but is inversely proportional to vehicle spacing, i.e.,

$$\lambda = \lambda_1 / S(t) = \lambda_1 / [x_n(t) - x_{n+1}(t)] \qquad (4.37)$$

where λ_1 is a new parameter - assumed to be a constant and which shall be referred to as the sensitivity coefficient. Using Equation 4.37 in Equation 4.32, our car following equation is:

$$\ddot{x}_{n+1}(t+\mathrm{T}) = \frac{\lambda_1}{[x_n(t) - x_{n+1}(t)]}[\dot{x}_n(t) - \dot{x}_{n+1}(t)] \qquad (4.38)$$

for $n = 1,2,3,...$

As before, by assuming the parameters are such that the traffic stream is stable, this equation can be integrated yielding the steady-state relation for speed and concentration:

$$u = \lambda_1 \ln (k_j / k) \qquad (4.39)$$

and for steady-state flow and concentration:

$$q = \lambda_1 k ln(k_j / k) \qquad (4.40)$$

where again it is assumed that for u=0, the spacing is equal to an effective vehicle length, $L = k^{-1}$. These relations for steady-state flow are identical to those obtained from considering the traffic stream to be approximated by a continuous compressible fluid (see Chapter 5) with the property that disturbances are propagated with a constant speed with respect to the moving medium (Greenberg 1959). For our non-linear car following equation, infinitesimal disturbances are propagated with speed λ_1 . This is consistent with the earlier discussion regarding the speed of propagation of a disturbance per vehicle pair.

It can be shown that if the propagation time, T_0, is directly proportional to spacing (i.e., $T_0 \approx S$), Equations 4.39 and 4.40 are obtained where the constant ratio S / T_o is identified as the constant λ_1.

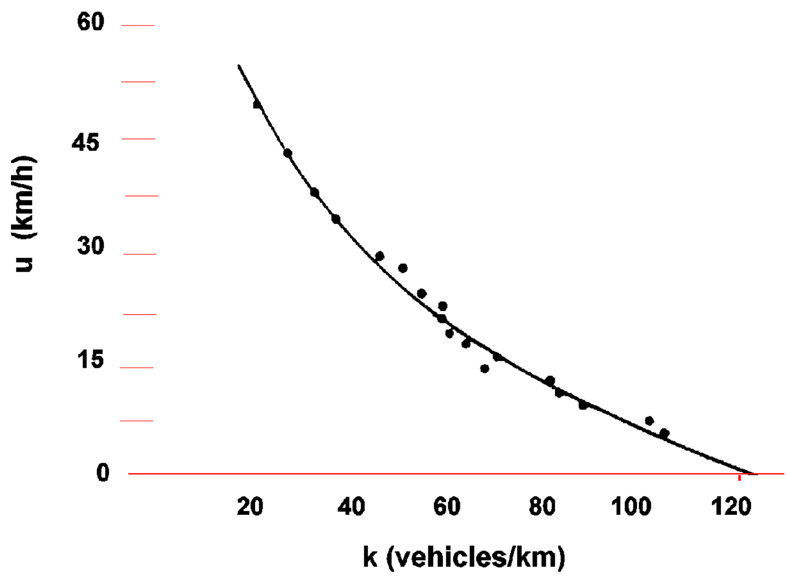
These two approaches are not analogous. In the fluid analogy case, the speed-spacing relationship is 'followed' at <u>every instant</u> before, during, and after a disturbance. In the case of car following during the transition phase, the speed-spacing, and therefore the flow-concentration relationship, does not describe the state of the traffic stream.

A solution to any particular set of equations for the motion of a traffic stream specifies departures from the steady-state. This is not the case for simple headway models or hydro-dynamical approaches to single-lane traffic flow because in these cases any small speed change, once the disturbance arrives, each vehicle instantaneously relaxes to the new speed, at the 'proper' spacing.

This emphasizes the shortcoming of these alternate approaches. They cannot take into account the behavioral and physical aspects of disturbances. In the case of car following models, the initial phase of a disturbance arrives at the *n*th vehicle downstream from the vehicle initiating the speed change at a time $(n\text{-}1)T$ seconds after the onset of the fluctuation. The time it takes vehicles to reach the changed speed depends on the parameter λ, for the linear model, and λ_1, for the non-linear model, subject to the restriction that $\lambda^{-1} > T$ or $\lambda_1 < S/T$, respectively.

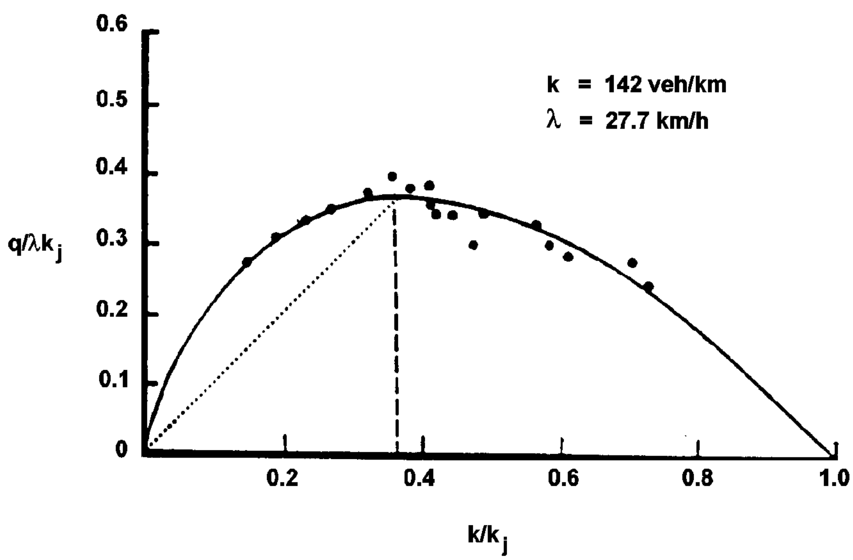
These restrictions assure that the signal speed can never precede the initial phase speed of a disturbance. For the linear case, the restriction is more than satisfied for an asymptotic stable traffic stream. For small speed changes, it is also satisfied for the non-linear model by assuming that the stability criteria results for the linear case yields a bound for the stability in the non-linear case. Hence, the inequality $\lambda T/S^*<0.5$ provides a sufficient stability condition for the non-linear case, where S^* is the minimum spacing occurring during a transition from one steady-state to another.

Before discussing a more general form for the sensitivity coefficient (i.e., Equation 4.37), the same reported data (Greenberg 1959) plotted in Figures 4.9 and 4.10 are graphed in Figures 4.11 and 4.12 together with the steady-state relations (Equations 4.39 and 4.40 obtained from the non-linear model, Equation 4.38). The fit of the data to the steady-state relation via the method of "least squares" is good and the resulting values for λ_1 and k_j are 27.7 km/h and 142 veh/km, respectively. Assuming that this data is a representative sample of this facility's traffic, the value of 27.7 km/h is an estimate not only of the sensitivity coefficient for the non-linear car following model but it is the 'characteristic speed' for the roadway under consideration (i.e., the speed of the traffic stream which maximizes the flow).



Note: The curve corresponds to a "least squares" fit of Equation 4.39 to the data (Greenberg 1959).

Figure 4.11
Speed Versus Vehicle Concentration (Gazis et al. 1959).



Note: The curve corresponds to Equation 4.40 where parameters are those from the "fit" obtained in Figure 4.11.

Figure 4.12
Normalized Flow Versus Normalized Vehicle Concentration (Edie et al. 1963).

The corresponding vehicle concentration at maximum flow, i.e., when $u = \lambda_1$, is $e^{-1} k_j$. This predicts a roadway capacity of $\lambda_1 e^{-1} k_j$ of about $\approx$1400 veh/h for the Lincoln Tunnel. A noted undesirable property of Equation 4.40 is that the tangent dq/dt is infinite at $k = 0$, whereas a linear relation between flow and concentration would more accurately describe traffic near zero concentration. This is not a serious defect in the model since car following models are not applicable for low concentrations where spacings are large and the coupling between vehicles are weak. However, this property of the model did suggest the following alternative form (Edie 1961) for the gain factor,

$$\lambda = \lambda_2 \dot{x}_{n+1}(t+\mathrm{T})/[x_n(t)-x_{n+1}(t)]^2$$

This leads to the following expression for a car following model:

$$\ddot{x}_{n+1}(t+\mathrm{T}) = \frac{\lambda_2 \dot{x}_{n+1}(t+\mathrm{T})}{[x_n(t)-x_{n+1}(t)]^2}[\dot{x}_n(t)-\dot{x}_{n+1}(t)] \qquad (4.41)$$

As before, this can be integrated giving the following steady-state equations:

$$U = U_f e^{-k/k_m} \qquad (4.42)$$

and

$$q = U_f k e^{-k/k_m} \qquad (4.43)$$

where U_f is the "free mean speed", i.e., the speed of the traffic stream near zero concentration and k_m is the concentration when the flow is a maximum. In this case the sensitivity coefficient, λ_2 can be identified as k_m^{-1}. The speed at optimal flow is $e^{-1}U_f$ which, as before, corresponds to the speed of propagation of a disturbance with respect to the moving traffic stream. This model predicts a finite speed, U_f, near zero concentration.

Ideally, this speed concentration relation should be translated to the right in order to more completely take into account observations that the speed of the traffic stream is independent of vehicle concentration for low concentrations, .i.e.

$$U = U_f \qquad for \qquad 0 \le k \le k_f \qquad (4.44)$$

and

$$U = U_f \exp - \left[\frac{k - k_f}{k_m}\right] \qquad (4.45)$$

where k_f corresponds to a concentration where vehicle to vehicle interactions begin to take place so that the stream speed begins to decrease with increasing concentration. Assuming that interactions take place at a spacing of about 120 m, k_f would have a value of about 8 veh/km. A "kink" of this kind was introduced into a linear model for the speed concentration relationship (Greenshields 1935).

Greenshields' empirical model for a speed-concentration relation is given by

$$U = U_f(1-k/k_j) \qquad (4.46)$$

where U_f is a "free mean speed" and k_j is the jam concentration.

It is of interest to question what car following model would correspond to the above steady-state equations as expressed by Equation 4.46. The particular model can be derived in the following elementary way (Gazis et al. 1961). Equation 4.46 is rewritten as

$$U = U_f(1-L/S) \qquad (4.47)$$

Differentiating both sides with respect to time obtains

$$\dot{U} = (U_f L/S^2)\dot{S} \qquad (4.48)$$

which after introduction of a time lag is for the $(n+1)$ vehicle:

$$\ddot{x}_{n+1}(t+\mathrm{T}) = \frac{U_f L}{[x_n(t)-x_{n+1}(t)]^2}[\dot{x}_n(t)-\dot{x}_{n+1}(t)] \qquad (4.49)$$

The gain factor is:

$$\frac{U_f L}{[x_n(t) - x_{n+1}(t)]^2} \tag{4.50}$$

The above procedure demonstrates an alternate technique at arriving at stimulus response equations from relatively elementary considerations. This method was used to develop early car following models (Reuschel 1950; Pipes 1951). The technique does pre-suppose that a speed-spacing relation reflects detailed psycho-physical aspects of how one vehicle follows another. To summarize the car-following equation considered, we have:

$$\ddot{x}_{n+1}(t+\mathrm{T}) = \lambda[\dot{x}_n(t) - \dot{x}_{n+1}(t)] \tag{4.51}$$

where the factor, λ, is assumed to be given by the following:

- A constant, $\lambda = \lambda_0$;
- A term inversely proportional to the spacing, $\lambda = \lambda_1/S$;
- A term proportional to the speed and inversely proportional to the spacing squared, $\lambda = \lambda_2 U/S^2$; and
- A term inversely proportional to the spacing squared, $\lambda = \lambda_3 / S^2$.

These models can be considered to be special cases of a more general expression for the gain factor, namely:

$$\lambda = a_{\ell,m} \dot{x}^{m}_{n+1}(t+\mathrm{T})/[x_n(t) - x_{n+1}(t)]^{\ell} \tag{4.52}$$

where $a_{\ell,m}$ is a constant to be determined experimentally. Model specification is to be determined on the basis of the degree to which it presents a consistent description of actual traffic phenomena. Equations 4.51 and 4.52 provide a relatively broad framework in so far as steady-state phenomena is concerned (Gazis et al. 1961).

Using these equations and integrating over time we have

$$f_m(U) = a \cdot f_l(S) + b \tag{4.53}$$

where, as before, U is the steady-state speed of the traffic stream, S is the steady-state spacing, and a and b are appropriate constants consistent with the physical restrictions and where $f_p(x)$, ($p = m$ or ℓ), is given by

$$f_p(x) = x^{1-p} \tag{4.54}$$

for $p \neq 1$ and

$$f_p(x) = \ell n x \tag{4.55}$$

for $p = 1$. The integration constant b is related to the "free mean speed" or the "jam concentration" depending on the specific values of m and ℓ. For $m > 1$, $\ell \neq 1$, or $m = 1$, $\ell > 1$

$$b = f_m(U_f) \tag{4.56}$$

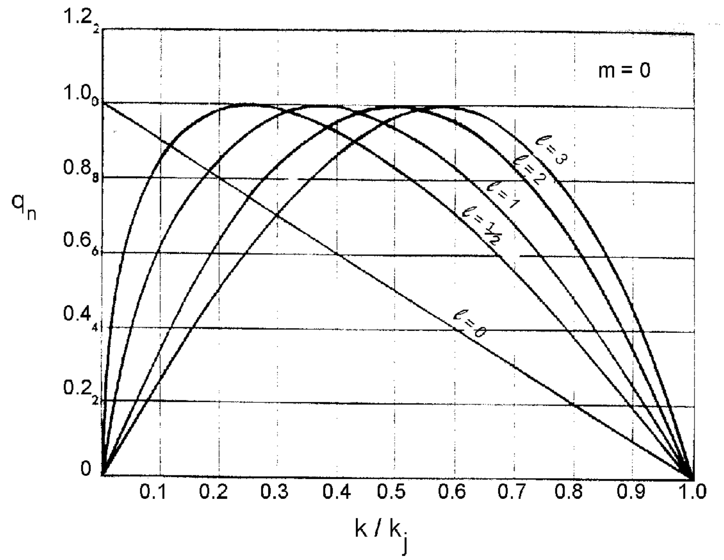
and

$$b = -a f_l(L) \tag{4.57}$$

for all other combinations of m and ℓ, except $\ell < 1$ and $m = 1$.

For those cases where $\ell < 1$ and $m = 1$ it is not possible to satisfy either the boundary condition at $k = 0$ or k_j and the integration constant can be assigned arbitrarily, e.g., at k_m, the concentration at maximum flow or more appropriately at some 'critical' concentration near the boundary condition of a "free speed" determined by the "kink" in speed-concentration data for the particular facility being modeled. The relationship between k_m and k_j is a characteristic of the particular functional or model being used to describe traffic flow of the facility being studied and not the physical phenomenon involved. For example, for the two models given by $\ell = 1$, $m = 0$, and $\ell = 2$, m = 0, maximum flow occurs at a concentration of $e^{-l} k_j$ and $k_j / 2$, respectively. Such a result is not physically unrealistic. Physically the question is whether or not the measured value of q_{max} occurs at or near the numerical value of these terms, i.e., $k_m = e^{-1} k_j$ or $k_j/2$ for the two examples cited.

Using Equations 4.53, 4.54, 4.55, 4.56, 4.57, and the definition of steady-state flow, we can obtain the relationships between speed, concentration, and flow. Several examples have been given above. Figures 4.13 and 4.14 contain these and additional examples of flow versus concentration relations for various



Note: Normalized flow versus normalized concentration corresponding to the steady-state solution of Equations 4.51 and 4.52 for m=1 and various values of ℓ.

Figure 4.13
Normalized Flow Versus Normalized Concentration (Gazis et al. 1963).

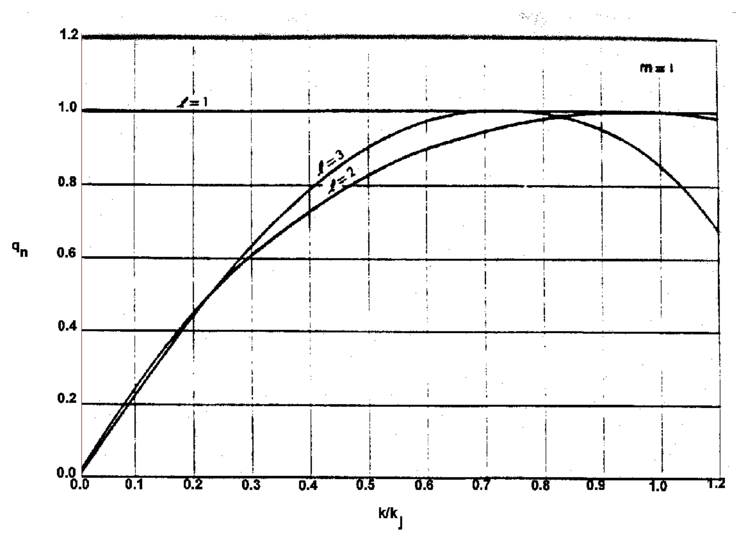


Figure 4.14
Normalized Flow versus Normalized Concentration Corresponding to the Steady-State Solution of Equations 4.51 and 4.52 for m=1 and Various Values of ℓ (Gazis 1963).

values of ℓ and m. These flow curves are normalized by letting $q_n = q/q_{max}$, and $k_n = k/k_j$.

It can be seen from these figures that most of the models shown here reflect the general type of flow diagram required to agree with the qualitative descriptions of steady-state flow. The spectrum of models provided are capable of fitting data like that shown in Figure 4.9 so long as a suitable choice of the parameters is made.

The generalized expression for car following models, Equations 4.51 and 4.52, has also been examined for non-integral values for m and ℓ (May and Keller 1967). Fittingdata obtained on the Eisenhower Expressway in Chicago they proposed a model with $m = 0.8$ and $\ell = 2.8$. Various values for m and ℓ can be identified in the early work on steady-state flow and car following .

The case $m = 0$, $\ell = 0$ equates to the "simple" linear car following model. The case $m = 0$, $\ell = 2$ can be identified with a model developed from photographic observations of traffic flow made in 1934 (Greenshields 1935). This model can also be developed considering the perceptual factors that are related to the car following task (Pipes and Wojcik 1968; Fox and Lehman 1967; Michaels 1963). As was discussed earlier, the case for $m = 0$, $\ell = 1$ generates a steady-state relation that can be developed bya fluid flow analogy to traffic (Greenberg 1959) and led to the reexamination of car following experiments and the hypothesis that drivers do not have a constant gain factor to a given relative-speed stimulus but rather that it varies inversely with the vehicle spacing, i.e., $m= 0$, $\ell =1$ (Herman et al. 1959). A generalized equation for steady-state flow (Drew 1965) and subsequently tested on the Gulf Freeway in Houston, Texas led to a model where $m = 0$ and $\ell = 3/2$.

As noted earlier, consideration of a "free-speed" near low concentrations led to the proposal and subsequent testing of the model $m = 1$, $\ell = 2$ (Edie 1961). Yet another model, $m = 1$, $\ell = 3$ resulted from analysis of data obtained on the Eisenhower Expressway in Chicago (Drake et al. 1967). Further analysis of this model together with observations suggest that the sensitivity coefficient may take on different values above a lane flow of about 1,800 vehicles/hr (May and Keller 1967).

4.4 Experiments And Observations

This section is devoted to the presentation and discussion of experiments that have been carried out in an effort to ascertain whether car following models approximate single lane traffic characteristics. These experiments are organized into two distinct categories.

The first of these is concerned with comparisons between car following models and detailed measurements of the variables involved in the driving situation where one vehicle follows another on an empty roadway. These comparisons lead to a quantitative measure of car following model estimates for the specific parameters involved for the traffic facility and vehicle type used.

The second category of experiments are those concerned with the measurement of macroscopic flow characteristics: the study of speed, concentration, flow and their inter-relationships for vehicle platoons and traffic environments where traffic is channeled in a single lane. In particular, the degree to which this type of data fits the analytical relationships that have been derived from car following models for steady-state flow are examined.

Finally, the degree to which any specific model of the type examined in the previous section is capable of representing a consistent framework from both the microscopic and macroscopic viewpoints is examined.

4.4.1 Car Following Experiments

The first experiments which attempted to make a preliminary evaluation of the linear car following model were performed a number of decades ago (Chandler et al. 1958; Kometani and Sasaki 1958). In subsequent years a number of different tests with varying objectives were performed using two vehicles, three vehicles, and buses. Most of these tests were conducted on test track facilities and in vehicular tunnels.

In these experiments, inter-vehicle spacing, relative speed, speed of the following vehicle, and acceleration of the following vehicles were recorded simultaneously together with a clock signal to assure synchronization of each variable with every other.

These car following experiments are divided into six specific categories as follows:

1) *Preliminary Test Track Experiments*. The first experiments in car following were performed by (Chandler et al. 1958) and were carried out in order to obtain estimates of the parameters in the linear car following model and to obtain a preliminary evaluation of this model. Eight male drivers participated in the study which was conducted on a one-mile test track facility.

2) *Vehicular Tunnel Experiments.* To further establish the validity of car following models and to establish estimates, the parameters involved in real operating environments where the traffic flow characteristics were well known, a series of experiments were carried out in the Lincoln, Holland, and Queens Mid-Town Tunnels of New York City. Ten different drivers were used in collecting 30 test runs.

3) *Bus Following Experiments.* A series of experiments were performed to determine whether the dynamical characteristics of a traffic stream changes when it is composed of vehicles whose performance measures are significantly different than those of an automobile. They were also performed to determine the validity and measure parameters of car following models when applied to heavy vehicles. Using a 4 kilometer test track facility and 53-passenger city buses, 22 drivers were studied.

4) *Three Car Experiments.* A series of experiments were performed to determine the effect on driver behavior when there is an opportunity for next-nearest following and of following the vehicle directly ahead. The degree to which a driver uses the information that might be obtained from a vehicle two ahead was also examined. The relative spacings and the relative speeds between the first and third vehicles and the second and third vehicles together with the speed and acceleration of the third vehicle were recorded.

5) *Miscellaneous Experiments.* Several additional car following experiments have been performed and reported on as follows:

a) *Kometani and Sasaki Experiments.* Kometani and Sasaki conducted and reported on a series of experiments that were performed to evaluate the effect of an additional term in the linear car following equation. This term is related to the acceleration of the lead vehicle. In particular, they investigated a model rewritten here in the following form:

$$\ddot{x}_{n+1}(t+\mathrm{T}) = \lambda[\dot{x}_n(t) - \dot{x}_{n+1}(t)] + \gamma\ddot{x}_n(t) \qquad (4.58)$$

This equation attempts to take into account a particular driving phenomenon, where the driver in a particular state realizes that he should maintain a non-zero acceleration even though the relative speed has been reduced to zero or near zero. This situation was observed in several cases in tests carried out in the vehicular tunnels - particularly when vehicles were coming to a stop. Equation 4.58 above allows for a non-zero acceleration when the relative speed is zero. A value of γ near one would indicate an attempt to nearly match the acceleration of the lead driver for such cases. This does not imply that drivers are good estimators of relative acceleration. The conjecture here is that by pursuing the task where the lead driver is undergoing a constant or near constant acceleration maneuver, the driver becomes aware of this qualitatively after nullifying out relative speed - and thereby shifts the frame of reference. Such cases have been incorporated into models simulating the behavior of bottlenecks in tunnel traffic (Helly 1959).

b) *Experiments of Forbes et al.* Several experiments using three vehicle platoons were reported by Forbes et al. (1957). Here a lead vehicle was driven by one of the experimenters while the second and third vehicles were driven by subjects. At predetermined locations along the roadway relatively severe acceleration maneuvers were executed by the lead vehicle. Photographic equipment recorded these events with respect to this moving reference together with speed and time. From these recordings speeds and spacings were calculated as a function of time. These investigators did not fit this data to car following models. However, a partial set of this data was fitted to vehicle following models by another investigator (Helly 1959). This latter set consisted of six tests in all, four in the Lincoln Tunnel and two on an open roadway.

c) *Ohio State Experiments.* Two different sets of experiments have been conducted at Ohio State University. In the first set a series of subjects have been studied using a car following simulator (Todosiev 1963). An integral part of the simulator is an analog computer which could program the lead vehicle for many different driving tasks. The computer could also simulate the performance characteristics of different following vehicles. These experiments were directed toward understanding the manner in which the following vehicle behaves when the lead vehicle moves with constant speed and the measurement of driver thresholds for changes in spacing, relative speed, and acceleration. The second set of experiments were conducted on a level two-lane state highway operating at low traffic concentrations (Hankin and Rockwell 1967). In these experiments the purpose was "to develop an empirically based model of car following which would predict a following car's acceleration and change in acceleration as a function of observed dynamic relationships with the lead car." As in the earlier car following experiments, spacing and relative speed were recorded as well as speed and acceleration of the following vehicle.

d) *Studies by Constantine and Young.* These studies were carried out using motorists in England and a photographic system to record the data (Constantine and Young 1967). The experiments are interesting from the vantage point that they also incorporated a second photographic system mounted in the following vehicle and directed to the rear so that two sets of car following data could be obtained simultaneously. The latter set collected information on an unsuspecting motorist. Although accuracy is not sufficient, such a system holds promise.

4.4.1.1 Analysis of Car Following Experiments

The analysis of recorded information from a car following experiment is generally made by reducing the data to numerical values at equal time intervals. Then, a correlation analysis is carried out using the linear car following model to obtain estimates of the two parameters, λ and T. With the data in discrete form, the time lag ,T, also takes on discrete values. The time lag or response time associated with a given driver is one for which the correlation coefficient is a maximum and typically falls in the range of 0.85 to 0.95.

The results from the preliminary experiments (Chandler et al. 1958) are summarized in Table 4.1 where the estimates are given for λ, their product; $C = \lambda T$, the boundary value for asymptotic stability; average spacing, $< S >$; and average speed, $< U >$. The average value of the gain factor is 0.368 sec^{-1}. The average value of λT is close to 0.5, the asymptotic stability boundary limit.

Table 4.1 Results from Car-Following Experiment

Driver	λ	$<U>$	$<S>$	λT
1	0.74 sec^{-1}	19.8 m/sec	36 m	1.04
2	0.44	16	36.7	0.44
3	0.34	20.5	38.1	1.52
4	0.32	22.2	34.8	0.48
5	0.38	16.8	26.7	0.65
6	0.17	18.1	61.1	0.19
7	0.32	18.1	55.7	0.72
8	0.23	18.7	43.1	0.47

Using the values for λ and the average spacing $<S>$ obtained for each subject a value of 12.1 m/sec (44.1 km/h) is obtained for an estimate of the constant $a_{1,0}$ (Herman and Potts 1959). This latter estimate compares the value λ for each driver with that driver's average spacing, $<S>$, since each driver is in somewhat different driving state. This is illustrated in Figure 4.15. This approach attempts to take into account the differences in the estimates for the gain factor λ or $a_{0,0}$, obtained for different drivers by attributing these differences to the differences in their respective average spacing. An alternate and more direct approach carries out the correlation analysis for this model using an equation which is the discrete form of Equation 4.38 to obtain a direct estimate of the dependence of the gain factor on spacing, $S(t)$.

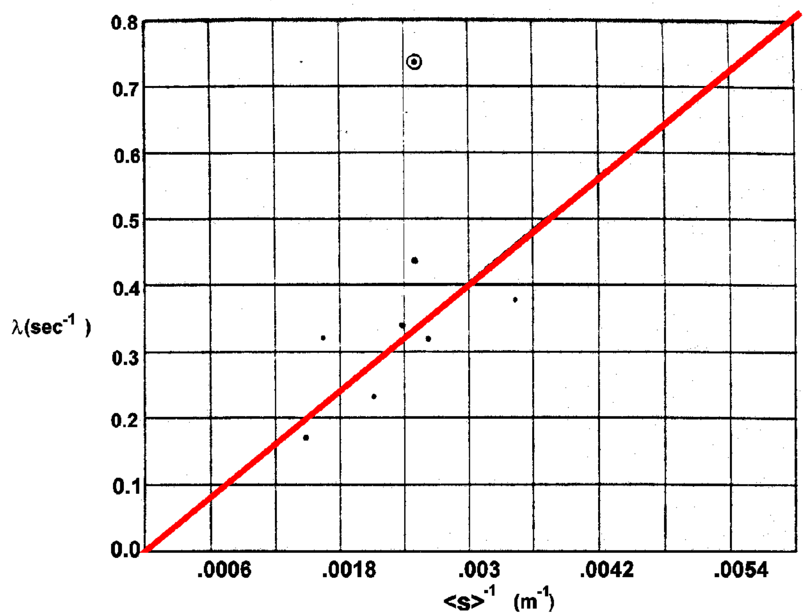


Figure 4.15
Sensitivity Coefficient Versus the Reciprocal of the Average Vehicle Spacing (Gazis et al. 1959).

Vehicular Tunnel Experiments. Vehicular tunnels usually have roadbeds that are limited to two lanes, one per direction. Accordingly, they consist of single-lane traffic where passing is prohibited. In order to investigate the reasonableness of the non-linear model a series of tunnel experiments were conducted. Thirty test runs in all were conducted: sixteen in the Lincoln Tunnel, ten in the Holland Tunnel and four in the Queens Mid-Town Tunnel. Initially, values of the parameters for the linear model were obtained, i.e., $\lambda = a_{0,0}$ and T. These results are shown in Figure 4.16 where the gain factor, $\lambda = a_{0,0}$ versus the time lag, T, for all of the test runs carried out in the vehicular tunnels. The solid curve divides the domain of this two parameter field into asymptotically stable and unstable regions.

It is of interest to note that in Figure 4.16 that many of the drivers fall into the unstable region and that there are drivers who have relatively large gain factors and time lags. Drivers with relatively slow responses tend to compensatingly have fast movement times and tend to apply larger brake pedal forces resulting in larger decelerations.

Such drivers have been identified, statistically, as being involved more frequently in "struck-from-behind accidents" (Babarik 1968; Brill 1972). Figures 4.17 and 4.18 graph the gain factor versus the reciprocal of the average vehicle spacing for the tests conducted in the Lincoln and Holland tunnels, respectively. Figure 4.17, the gain factor, λ, versus the reciprocal of the average spacing for the Holland Tunnel tests. The straight line is a "least-squares" fit through the origin. The slope, which is an estimate of $a_{1,0}$ and equals 29.21 km/h. Figure 4.18 graphs the gain factor, λ, versus the reciprocal of the average spacing for the Lincoln Tunnel tests. The straight line is a "least-squares" fit through the origin. These results yield characteristic speeds, $a_{1,0}$, which are within ± 3km/h for these two similar facilities. Yet these small numeric differences for the parameter $a_{1,0}$ properly reflect known differences in the macroscopic traffic flow characteristics of these facilities.

The analysis was also performed using these test data and the non-linear reciprocal spacing model. The results are not strikingly different (Rothery 1968). Spacing does not change significantly within any one test run to provide a sensitive measure of the dependency of the gain factor on inter-vehicular spacing for any given driver (See Table 4.2). Where the variation in spacings were relatively large (e.g., runs 3, 11, 13, and 14) the results tend to support the spacing dependent model. This time-dependent analysis has also been performed for seven additional functions for the gain factor for the same fourteen

Table 4.2
Comparison of the Maximum Correlations obtained for the Linear and Reciprocal Spacing Models for the Fourteen Lincoln Tunnel Test Runs

Number	$r_{0,0}$	$r_{l,0}$	<*S*> (m)	σ*S* (m)
1	0.686	0.459	13.4	4.2
2	0.878	0.843	15.5	3.9
3	0.77	0.778	20.6	5.9
4	0.793	0.748	10.6	2.9
5	0.831	0.862	12.3	3.9
6	0.72	0.709	13.5	2.1
7	0.64	0.678	5.5	3.2

Number	$r_{0,0}$	$r_{l,0}$	<*S*> (m)	σ*S* (m)
8	0.865	0.881	19.9	3.4
9	0.728	0.734	7.6	1.8
10	0.898	0.898	10.7	2.3
11	0.89	0.966	26.2	6.2
12	0.846	0.835	18.5	1.3
13	0.909	0.928	18.7	8.8
14	0.761	0.79	46.1	17.6

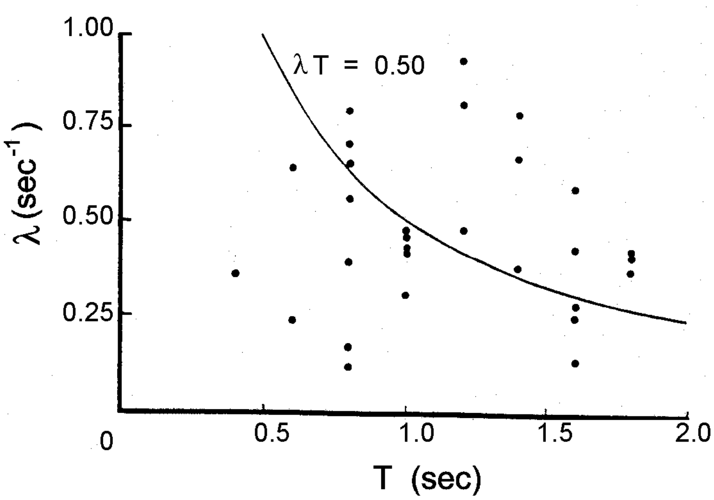
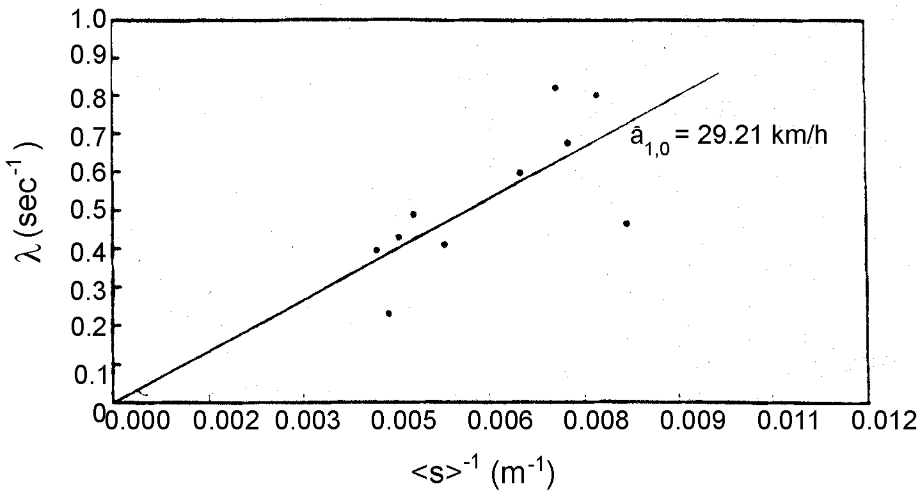
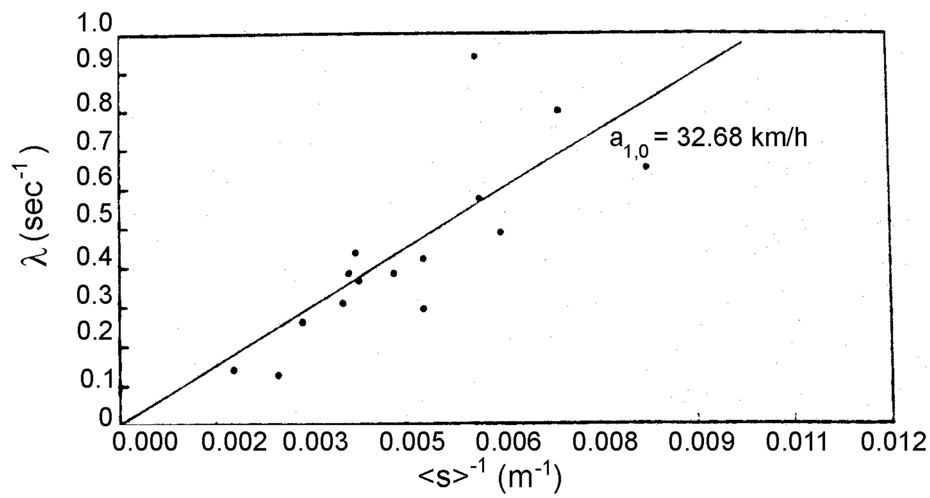


Figure 4.16
Gain Factor, λ, Versus the Time Lag, T, for All of the Test Runs (Rothery 1968).



Note: The straight line is a "least-squares" fit through the origin. The slope, which is an estimate of $a_{1,0}$, equals 29.21 km/h.

Figure 4.17
Gain Factor, λ, Versus the Reciprocal of the
Average Spacing for Holland Tunnel Tests (Herman and Potts 1959).



Note: The straight line is a "least-squares" fit through the origin. The slope, is an estimate of $a_{1,0}$, equals 32.68 km/h.

Figure 4.18
Gain Factor, λ ,Versus the Reciprocal of the
Average Spacing for Lincoln Tunnel Tests (Herman and Potts 1959).

little difference from one model to the other. There are definite trends however. If one graphs the correlation coefficient for a given ℓ, say ℓ=1 versus m; 13 of the cases indicate the best fits are with m = 0 or 1. Three models tend to indicate marginal superiority; they are those given by (ℓ=2; m=1), (ℓ=1; m=0) and (ℓ=2; m=0).

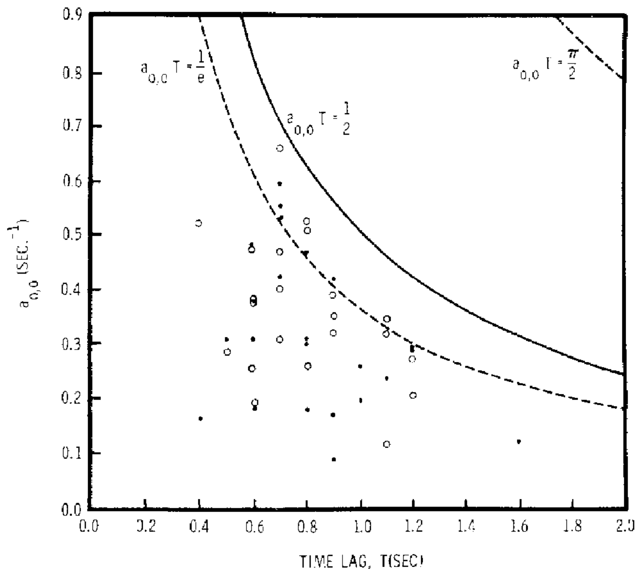
Bus Following Experiments. For each of the 22 drivers tested, the time dependent correlation analysis was carried out for the linear model (ℓ=0; m=0), the reciprocal spacing model (ℓ=1; m=0), and the speed, reciprocal-spacing-squared model (ℓ=2; m=1). Results similar to the Tunnel analysis were obtained: high correlations for almost all drivers and for each of the three models examined (Rothery et al. 1964).

The correlation analysis provided evidence for the reciprocal spacing effect with the correlation improved in about 75 percent of the cases when that factor is introduced and this model (ℓ=1; m=0) provided the best fit to the data. The principle results of the analysis are summarized in Figure 4.19 where the sensitivity coefficient $a_{0,0}$ versus the time lag, T, for the bus following experiments are shown. All of the data points obtained in these results fall in the asymptotically stable

region, whereas in the previous automobile experiments approximately half of the points fell into this region. In Figure 4.19, the sensitivity coefficient, $a_{0,0}$, versus the time lag, T, for the bus following experiments are shown. Some drivers are represented by more than one test. The circles are test runs by drivers who also participated in the ten bus platoon experiments. The solid curve divides the graph into regions of asymptotic stability and instability. The dashed lines are boundaries for the regions of local stability and instability.

Table 4.3
Maximum Correlation Comparison for Nine Models, $a_{\ell,m}$, for Fourteen Lincoln Tunnel Test Runs.

Driver	r(0,0)	r(1,-1)	r(1,0)	r(1,1)	r(1,2)	r(2,-1)	r(2,0)	r(2,1)	r(2,2)
1	0.686	0.408	0.459	0.693	0.721	0.310	0.693	0.584	0.690
2	0.878	0.770	0.843	0.847	0.746	0.719	0.847	0.827	0.766
3	0.770	0.757	0.778	0.786	0.784	0.726	0.786	0.784	0.797
4	0.793	0.730	0.748	0.803	0.801	0.685	0.801	0.786	0.808
5	0.831	0.826	0.862	0.727	0.577	0.805	0.728	0.784	0.624
6	0.720	0.665	0.709	0.721	0.709	0.660	0.720	0.713	0.712
7	0.640	0.470	0.678	0.742	0.691	0.455	0.745	0.774	0.718
8	0.865	0.845	0.881	0.899	0.862	0.818	0.890	0.903	0.907
9	0.728	0.642	0.734	0.773	0.752	0.641	0.773	0.769	0.759
10	0.898	0.890	0.898	0.893	0.866	0.881	0.892	0.778	0.865
11	0.890	0.952	0.966	0.921	0.854	0.883	0.921	0.971	0.940
12	0.846	0.823	0.835	0.835	0.823	0.793	0.835	0.821	0.821
13	0.909	0.906	0.923	0.935	0.927	0.860	0.935	0.928	0.936
14	0.761	0.790	0.790	0.771	0.731	0.737	0.772	0.783	0.775



Note: For bus following experiments - Some drivers are represented by more than one test. The circles are test runs by drivers who also participated in the ten bus platoons experiments. The solid curve divides the graph into regions of asymptotic stability and instability. The dashed lines are boundaries for the regions of local stability and instability.

Figure 4.19
Sensitivity Coefficient, $a_{0,0}$, Versus the Time Lag, T (Rothery et al. 1964).

The results of a limited amount of data taken in the rain suggest that drivers operate even more stably when confronted with wet road conditions. These results suggest that buses form a highly stable stream of traffic.

The time-independent analysis for the reciprocal-spacing model and the speed-reciprocal-spacing-squared model uses the time dependent sensitivity coefficient result, $a_{0,0}$, the average speed, $<U>$, and the average spacing, $<S>$, for eachof the car following test cases in order to form estimates of $a_{1,0}$ and $a_{2,1}$, i.e. by fitting

$$a_{0,0} = \frac{a_{1,0}}{<S>}$$

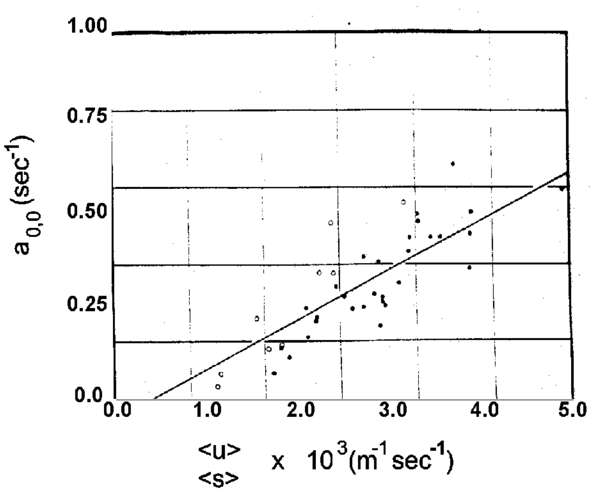
and

$$a_{0,0} = a_{2,1}\frac{<U>}{<S>^2}$$

respectively (Rothery et al. 1964).

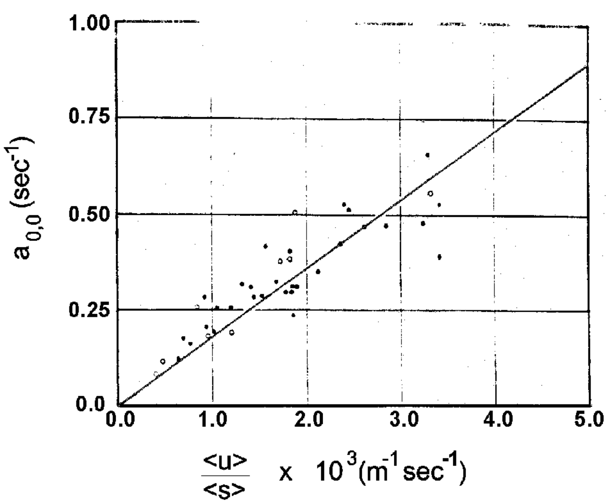
Figures 4.20 and 4.21 graph the values of $a_{0,0}$ for all test runs versus $<S>^{-1}$ and $<U>\ <S>^{-2}$, respectively. In Figure 4.20, the sensitivity coefficient versus the reciprocal of the average spacing for each bus following experiment, and the "least-squares" straight line are shown. The slope of this regression is an estimate of the reciprocal spacing sensitivity coefficient. The solid dots and circles are points for two different test runs.

In Figure 4.21, the sensitivity coefficient versus the ratio of the average speed to the square of the average spacing for each bus following experiment and the "least-square" straight line are shown. The slope of this regression is an estimate of the speed-reciprocal spacing squared sensitivity coefficient. The solid dots and circles are data points for two different test runs. The slope of the straight line in each of these figures give an estimate of their respective sensitivity coefficient for the sample population. For the reciprocal spacing model the results indicate an estimate for $a_{1,0} = 52.8 \pm .05$ m/sec. (58 ± 1.61 km/h) and for the speed-reciprocal spacing squared model $a_{2,1} = 54.3 \pm 1.86$ m. The errors are one standard deviation.



Note: The sensitivity coefficient versus the reciprocal of the average spacing for each bus following experiment. The least squares straight line is shown. The slope of this regression is an estimate of the reciprocal spacing sensitivity coefficient. The solid dots and circles are data points for two different test runs.

Figure 4.20
Sensitivity Coefficient Versus the Reciprocal of the Average Spacing (Rothery et al. 1964).



Note: The sensitivity coefficient versus the ratio of the average speed to the square of the average spacing for each bus following experiment. The least squares straight line is shown. The slope of this regression is an estimate of the speed-reciprocal spacing squared sensitivity coefficient. The solid dots and circles are data points for two different test runs.

Figure 4.21
Sensitivity Coefficient Versus the Ratio of the Average Speed (Rothery et al. 1964).

Three Car Experiments. These experiments were carried out in an effort to determine, if possible, the degree to which a driver is influenced by the vehicle two ahead, i.e., next nearest interactions (Herman and Rothery 1965). The data collected in these experiments are fitted to the car following model:

$$\ddot{x}_{n+2}(t+\mathrm{T}) = \lambda_1[\dot{x}_{n+1}(t) - \dot{x}_{n+2}(t)] + \lambda_2[\dot{x}_n(t) - \dot{x}_{n+2}(t)] \quad (4.61)$$

This equation is rewritten in the following form:

$$\ddot{x}_{n+2}(t+\mathrm{T}) = \lambda_l\left\{[\dot{x}_{n+1}(t) - \dot{x}_{n+2}(t)] + \delta[\dot{x}_n(t) - \dot{x}_{n+2}(t)]\right\} \quad (4.62)$$

where

$$\delta = \lambda_2/\lambda_1$$

A linear regression analysis is then conducted for specific values of the parameter δ. For the case $\delta = 0$ there is nearest neighbor coupling only and for $\delta >> 1$ there is next nearest

neighbor coupling only. Using eight specific values of δ (0, 0.25, 0.50, 1, 5, 10, 100, and ∞) and a mean response time of 1.6 sec, a maximum correlation was obtained when δ = 0. However, if response times are allowed to vary, modest improvements can be achieved in the correlations.

While next nearest neighbor couplings cannot be ruled out entirely from this study, the results indicate that there are no significant changes in the parameters or the correlations when following two vehicles and that the stimulus provided by the nearest neighbor vehicle, i.e., the 'lead' vehicle, is the most significant. Models incorporating next nearest neighbor interactions have also been used in simulation models (Fox and Lehman 1967). The influence of including such interactions in simulations are discussed in detail by those authors.

Miscellaneous Car Following Experiments. A brief discussion of the results of three additional vehicle following experiments are included here for completeness.

The experiments of Kometani and Sasaki (1958) were car following experiments where the lead vehicle's task was closely approximated by: "accelerate at a constant rate from a speed *u* to a speed *u'* and then decelerate at a constant rate from the speed *u'* to a speed *u*." This type of task is essentially 'closed' since the external situation remains constant. The task does not change appreciably from cycle to cycle. Accordingly, response times can be reduced and even canceled because of the cyclic nature of the task.

By the driver recognizing the periodic nature of the task or that the motion is sustained over a period of time (≈ 13 sec for the acceleration phase and ≈ 3 sec for the deceleration phase) the driver obtains what is to him/her advanced information. Accordingly, the analysis of these experiments resulted in short response times ≈ 0.73 sec for low speed (20-40 km/h.) tests and ≈ 0.54 sec for high speed (40-80 km/h.) tests. The results also produced significantly large gain factors. All of the values obtained for each of the drivers for λT, exceeded the asymptotic stability limit. Significantly better fits of the data can be made using a model which includes the acceleration of the lead vehicle (See Equation 4.58) relative to the linear model which does not contain such a term. This is not surprising, given the task of following the lead vehicle's motion as described above.

A partial set of the experiments conducted by Forbes et al. (1958) were examined by Helly (1959), who fitted test runs to the linear vehicle model, Equation 4.41, by varying λ and T to minimize the quantity:

$$\sum_{j=1}^{N} [\dot{x}_n^{Exp.}(j.\delta t) - \dot{x}_n^{Theor..}(j.\delta t)]^2 \quad (4.63)$$

where the data has been quantitized at fixed increments of δt, $N\delta t$ is the test run duration, $\dot{x}_n su^{\mathrm{Exp.}}(j.\delta t)$ is the experimentally measured values for the speed of the following vehicle at time $j\delta t$, and $\dot{x}_n^{\mathrm{Theor..}}(j.\delta t)$ is the theoretical estimate for the speed of the following vehicle as determined from the experimentally measured values of the acceleration of the following vehicle and the speed of the lead vehicle *using the linear model*. These results are summarized in Table 4.4.

Ohio State Simulation Studies. From a series of experiments conducted on the Ohio State simulator, a relatively simple car following model has been proposed for *steady-state* car following (Barbosa 1961). The model is based on the concept of driver thresholds and can be most easily described by means of a 'typical' recording of relative speed versus spacing as

Table 4.4 Results from Car Following Experiments

Driver #	T	a_{00}	r_{max}
1	1.0	0.7	0.86
2	0.5	1.3	0.96
3	0.6	0.8	0.91
4	0.5	1.0	0.87
5	0.7	1.1	0.96
6	0.5	1.0	0.86

shown in Figure 4.22. At point "1," it is postulated that the driver becomes aware that he is moving at a higher speed than the lead vehicle and makes the decision to decelerate in order to avoid having either the negative relative speed becoming too large or the spacing becoming too small. At point "A," after a time lag, the driver initiates this deceleration and reduces the relative speed to zero. Since drivers have a threshold level below which relative speed cannot be estimated with accuracy, the driver continues to decelerate until he becomes aware of a positive relative speed because it either exceeds the threshold at this spacing or because the change in spacing has exceeded its threshold level. At point "2," the driver makes the decision to accelerate in order not to drift away from the lead vehicle. This decision is executed at point "B" until point "3" is reach and the cycle is more or less repeated. It was found that the arcs, e.g., AB, BC, etc. are "approximately parabolic" implying that accelerations can be considered roughly to be constant. These accelerations have been studied in detail in order to obtain estimates of relative speed thresholds and how they vary with respect to inter-vehicle spacing and observation times (Todosiev 1963). The results are summarized in Figure 4.23. This driving task, following a lead vehicle traveling at constant speed, was also studied using automobiles in a driving situation so that the pertinent data could be collected in a closer-to-reality situation and then analyzed (Rothery 1968).

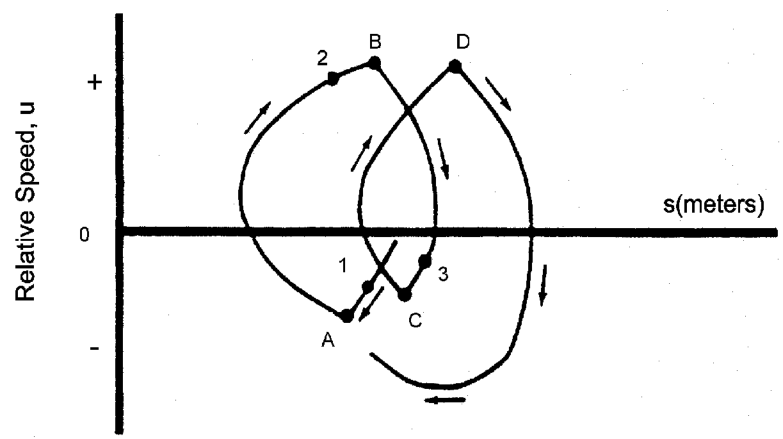


Figure 4.22
Relative Speed Versus Spacing (Rothery 1968).

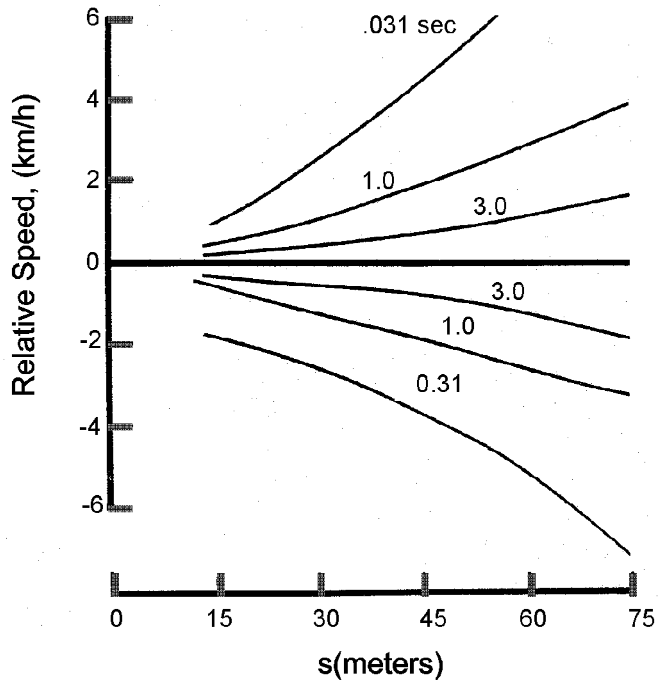


Figure 4.23
Relative Speed Thresholds Versus Inter-Vehicle Spacing for Various Values of the Observation Time. (Rothery 1968).

The interesting element in these latter results is that the character of the motion as exhibited in Figure 4.22 is much the same. However, the range of relative speeds at a given spacing that were recorded are much lower than those measured on the simulator. Of course the perceptual worlds in these two tests are considerably different. The three dimensional aspects of the test track experiment alone might provide sufficient additional cues to limit the subject variables in contrast to the two dimensional CRT screen presented to the 'driver' in the simulator. In any case, thresholds estimated in driving appear to be less than those measured via simulation.

Asymmetry Car Following Studies. One car following experiment was studied segment by segment using a model where the stimulus included terms proportional to deviations from the mean inter-vehicle spacing, deviations from the mean speed of the lead vehicle and deviations from the mean speed of the following car (Hankin and Rockwell 1967). An interesting result of the analysis of this model is that it implied an asymmetry in the response depending on whether the relative speed stimulus is positive or negative. This effect can be taken into account by rewriting our basic model as:

$$\dot{x}_{n+1}(t+\mathrm{T}) = \lambda_i[\dot{x}_n(t) - \dot{x}_{n+1}(t)] \qquad (4.64)$$

where $\lambda_i = \lambda_+$ or λ_- depending on whether the relative speed is greater or less than zero.

A reexamination of about forty vehicle following tests that were carried out on test tracks and in vehicular tunnels indicates, without exception, that such an asymmetry exists (Herman and Rothery 1965). The average value of λ_- is $\approx$10 percent greater than λ_+. The reason for this can partly be attributed to the fact that vehicles have considerably different capacities to accelerate and decelerate. Further, the degree of response is likely to be different for the situations where vehicles are separating compared to those where the spacing is decreasing. This effect creates a special difficulty with car following models as is discussed in the literature (Newell 1962; Newell 1965). One of the principal difficulties is that in a cyclic change in the lead vehicle's speed - accelerating up to a higher speed and then returning to the initial speed, the asymmetry in acceleration and deceleration of the following car prevents return to the original spacing. With *n* such cycles, the spacing continues to increase thereby creating a drifting apart of the vehicles. A relaxation process needs to be added to the models that allows for this asymmetry and also allows for the return to the correct spacing.

4.4.2 Macroscopic Observations: Single Lane Traffic

Several data collections on single lane traffic have been carried out with the specific purpose of generating a large sample from which accurate estimates of the macroscopic flow characteristics could be obtained. With such a data base, direct comparisons can be made with microscopic, car following estimates - particularly when the car following results are obtained on the same facility as the macroscopic data is collected. One of these data collections was carried out in the Holland Tunnel (Edie et al. 1963). The resulting macroscopic flow data for this 24,000 vehicle sample is shown in Table 4.5.

The data of Table 4.5 is also shown in graphical form, Figures 4.24 and 4.25 where speed versus concentration and flow versus concentration are shown, respectively. In Figure 4.24, speed versus vehicle concentration for data collected in the Holland Tunnel is shown where each data point represents a speed class of vehicles moving with the plotted speed ± 1.61 m/sec. In Figure 4.25, flow versus vehicle concentration is shown; the solid points are the flow values derived from the speed classes assuming steady-state conditions. (See Table 4.5 and Figure 4.24.) Also included in Figure 4.25 are one-minute average flow values shown as encircled points. (See Edie et al. 1963). Using this macroscopic data set, estimates for three sensitivity coefficients are estimated for the particular car following models that appear to be of significance. These are: $a_{1,0}$, $a_{2,1}$, and $a_{2,0}$. These are sometimes referred to as the Reciprocal Spacing Model, Edie's Model, and Greenshields' Model, respectively. The numerical values obtained are shown and compared with the microscopic estimates from car following experiments for these same parameters.

The associated units for these estimates are ft/sec, ft^2/sec, and miles/car, respectively. As illustrated in this table, excellent agreement is obtained with the reciprocal spacing model. How well these models fit the macroscopic data is shown in Figure 4.26, where the speed versus vehicle concentration data is graphed together with the curves corresponding to the steady-state speed-concentration relations for the various indicated models. The data appears in Figure 4.24 and 4.25.

The curves are least square estimates. All three models provide a good estimate of the characteristic speed (i.e., the speed at optimum flow, namely 19, 24, and 23 mi/h for the reciprocal spacing, reciprocal spacing squared, and speed reciprocal spacing squared models, respectively).

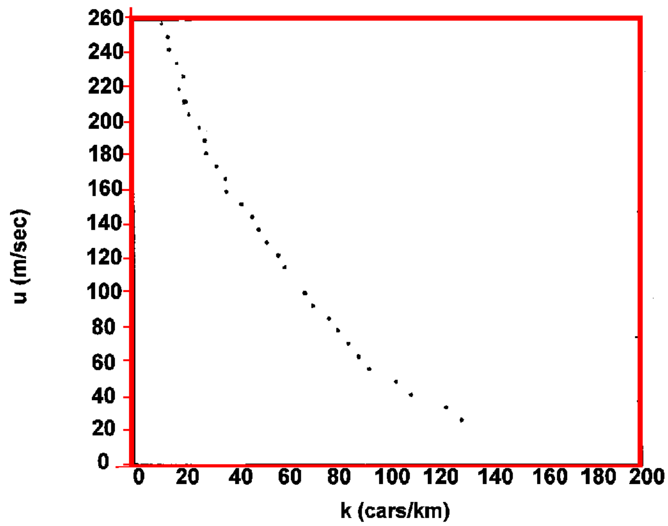
Edie's original motivation for suggesting the reciprocal spacing speed model was to attempt to describe low concentration, non-congested traffic. The key parameter in this model is the "mean free speed", i.e., the vehicular stream speed as the concentration goes to zero. The least squares estimate from the macroscopic data is 26.85 meters/second.

Edie also compared this model with the macroscopic data in the concentration range from zero to 56 vehicles/kilometer; the reciprocal spacing model was used for higher concentrations (Edie 1961). Of course, the two model fit is better than any one model fitted over the entire range, but marginally (Rothery 1968). Even though the improvement is marginal there is an apparent discontinuity in the derivative of the speed-concentration curve. This discontinuity is different than that which had previously been discussed in the literature. It had been suggested that there was an apparent break in the flow concentration curve near maximum flow where the flow drops suddenly (Edie and Foote 1958; 1960; 1961). That type of discontinuity suggests that the *u-k* curve is discontinuous.

However, the data shown in the above figures suggest that the curve is continuous and its derivative is not. If there is a discontinuity in the flow concentration relation near optimum flow it is considerably smaller for the Holland Tunnel than has been suggested for the Lincoln Tunnel. Nonetheless, the apparent discontinuity suggests that car following may be bimodal in character.

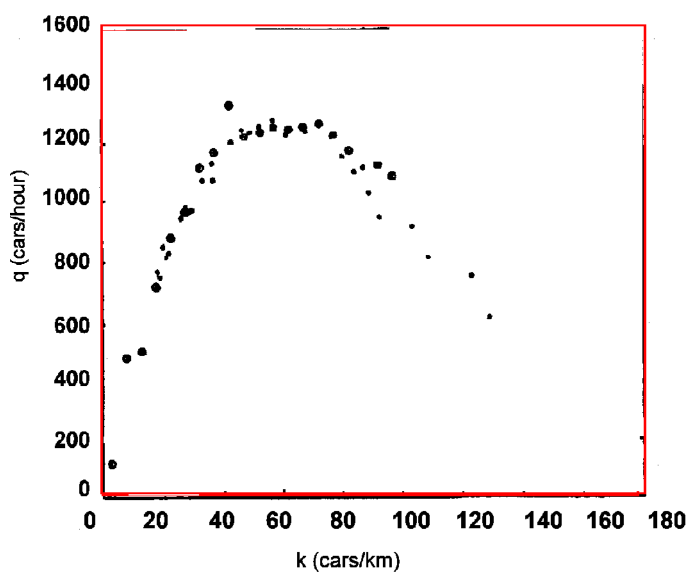
Table 4.5 Macroscopic Flow Data

Speed (m/sec)	Average Spacing (m)	Concentration (veh/km)	Number of Vehicles
2.1	12.3	80.1	22
2.7	12.9	76.5	58
3.3	14.6	67.6	98
3.9	15.3	64.3	125
4.5	17.1	57.6	196
5.1	17.8	55.2	293
5.7	18.8	52.6	436
6.3	19.7	50	656
6.9	20.5	48	865
7.5	22.5	43.8	1062
8.1	23.4	42	1267
8.7	25.4	38.8	1328
9.3	26.6	37	1273
9.9	27.7	35.5	1169
10.5	30	32.8	1096
11.1	32.2	30.6	1248
11.7	33.7	29.3	1280
12.3	33.8	26.8	1162
12.9`	43.2	22.8	1087
13.5	43	22.9	1252
14.1	47.4	20.8	1178
14.7	54.5	18.1	1218
15.3	56.2	17.5	1187
15.9	60.5	16.3	1135
16.5	71.5	13.8	837
17.1	75.1	13.1	569
17.7	84.7	11.6	478
18.3	77.3	12.7	291
18.9	88.4	11.1	231
19.5	100.4	9.8	169
20.1	102.7	9.6	55
20.7	120.5	8.1	56



Note: Each data point represents a speed class of vehicles moving with the plotted speed ± 1 ft/sec (See Table 4.4).

Figure 4.24
Speed Versus Vehicle Concentration (Edie et al. 1963).



Note: The solid points are the flow values derived from the speed classes assuming steady-state condition. Also included in Figure 4.25 are one minute average flow values shown as encircled points.

Figure 4.25
Flow Versus Vehicle Concentration (Edie et al. 1963).

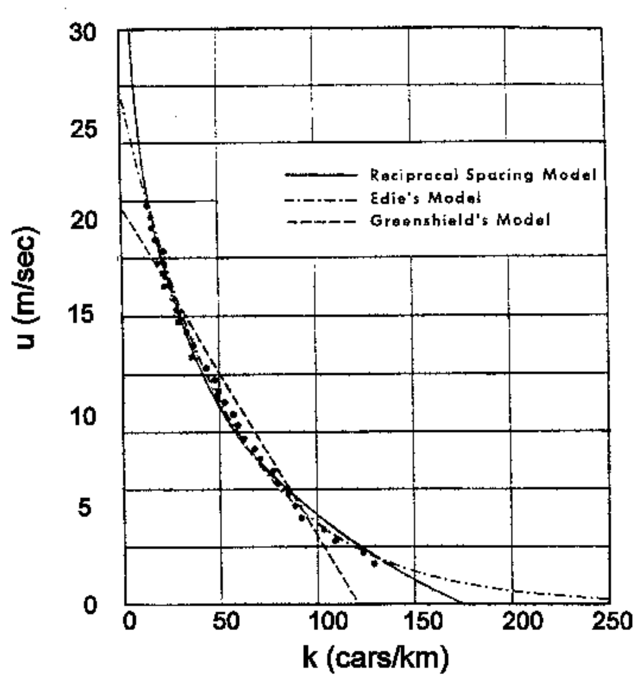


Figure 4.26
Speed Versus Vehicle Concentration (Rothery 1968).

Table 4.6
Parameter Comparison
(Holland Tunnel Data)

Parameters	Microscopic Estimates	Macroscopic Estimates
$a_{1,0}$	26.8	27.8
$a_{2,0}$	0.57	0.12
$a_{2,1}$	$(123)^{-1}$	$(54)^{-1}$

A totally different approach to modeling traffic flow variables which incorporates such discontinuities can be found in the literature. Navin (1986) and Hall (1987) have suggested that catastrophe theory (Thom 1975; Zeeman 1977) can be used as a vehicle for representing traffic relationships. Specifically, Navin followed the two regime approach proposed by Edie and cited above and first suggested that traffic relations can be represented using the cusp catastrophe. A serious attempt to apply such an approach to actual traffic data in order to represent flow variables without resorting to using two different expressions or two different sets of parameters to one expression has been made by Hall (1987). More recently, Acha-Daza and Hall (1994) have reported an analysis of freeway data using catastrophe theory which indicates that such an approach can effectively be applied to traffic flow. Macroscopic data has also been reported on single lane bus flow. Here platoons of ten buses were studied (Rothery et al. 1964).

Platoons of buses were used to quantify the steady-state stream properties and stability characteristics of single lane bus flow. Ideally, long chains of buses should be used in order to obtain the bulk properties of the traffic stream and minimize the end effects or eliminate this boundary effect by having the lead vehicle follow the last positioned vehicle in the platoon using a circular roadway. These latter type of experiments have been carried out at the Road Research Laboratory in England (Wardrop 1965; Franklin 1967).

In the platoon experiments, flow rates, vehicle concentration, and speed data were obtained. The average values for the speed and concentration data for the ten bus platoon are shown in Figure 4.28 together with the numerical value for the parameter

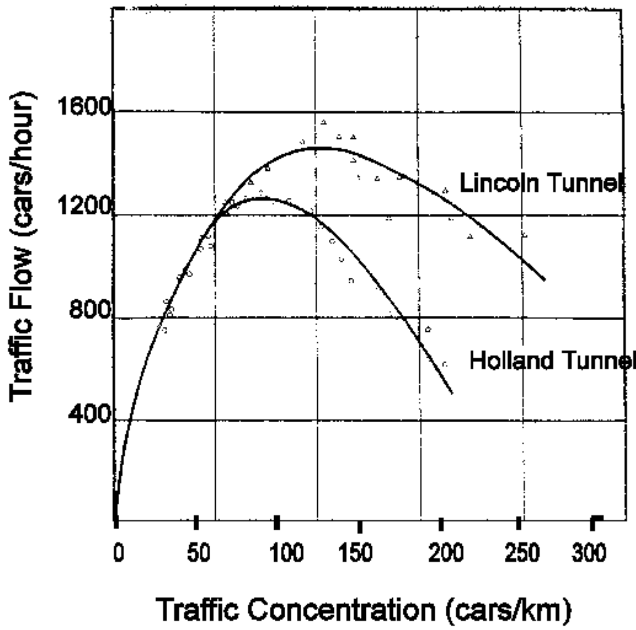


Figure 4.27
Flow Versus Concentration for the Lincoln and Holland Tunnels.

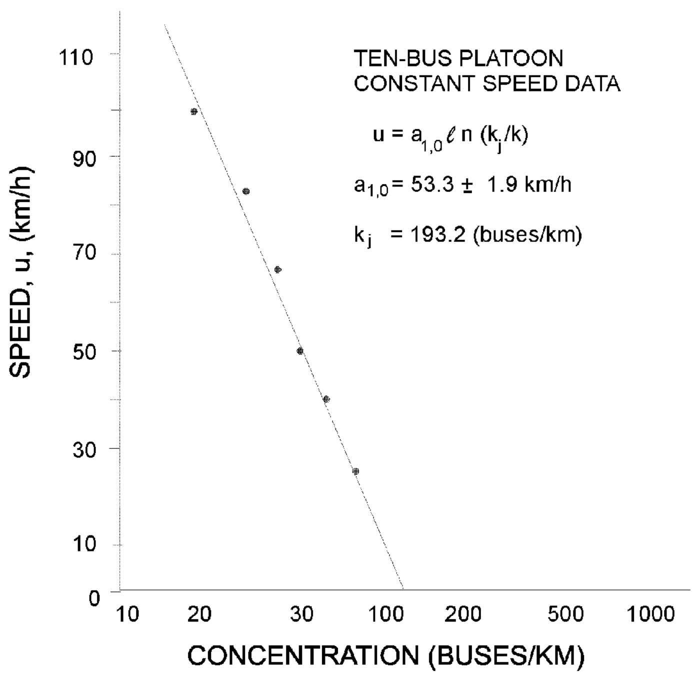


Figure 4.28
Average Speed Versus Concentration
for the Ten-Bus Platoon Steady-State Test Runs (Rothery 1968).

$a_{1,0}$ = 53 km/h which is to be compared to that obtained from the two bus following experiments discussed earlier namely, 58 km/h. Given these results, it is estimated that a single lane of standard size city buses is stable and has the capacity of over 65,000 seated passengers/hour. An independent check of this result has been reported (Hodgkins 1963). Headway times and speed of clusters of three or more buses on seven different highways distributed across the United States were measured and concluded that a maximum flow for buses would be approximately 1300 buses/hour and that this would occur at about 56 km/h.

4.5 Automated Car Following

All of the discussion in this chapter has been focused on manual car following, on what drivers do in following one or more other vehicles on a single lane of roadway. Paralleling these studies, research has also focused on developing controllers that would automatically mimic this task with specific target objectives in mind.

At the 1939 World's Fair, General Motors presented conceptually such a vision of automated highways where vehicles were controlled both longitudinally (car following) and laterally thereby freeing drivers to take on more leisurely activities as they moved at freeway speeds to their destinations. In the intervening years considerable effort has been extended towards the realization of this transportation concept. One prime motivation for such systems is that they are envisioned to provide more efficient utilization of facilities by increasing roadway capacity particularly in areas where constructing additional roadway lanes is undesirable and or impractical, and in addition, might improve safety levels. The concept of automated highways is one where vehicles would operate on both conventional roads under manual control and on specially instrumented guideways under automatic control. Here we are interested in automatic control of the car following task. Early research in this arena was conducted on both a theoretical and experimental basis and evaluated at General Motors Corporation (Gardels 1960; Morrison 1961; Flory et al. 1962), Ohio State University (Fenton 1968; Bender and Fenton 1969; Benton et al. 1971; Bender and Fenton 1970), Japan Governmental Mechanical Laboratory (Oshima et al. 1965), the Transportation Road Research Laboratory (Giles and Martin 1961; Cardew 1970), Ford Motor Corporation (Cro and Parker 1970) and the Japanese Automobile Research Institute (Ito 1973). During the past several decades three principal research studies in this arena stand out: a systems study of automated highway systems conducted at General Motors from 1971-1981, a long-range program on numerous aspects of automated highways conducted at The Ohio State University from 1964-1980, and the Program on Advanced Technology for the Highway (PATH) at the University of California, Berkeley from about 1976 to the present. Three overviews and detailed references to milestones of these programs can be found in the literature: Bender (1990), Fenton and Mayhan (1990), and Shladover et al. (1990), respectively.

The car following elements in these studies are focused on developing controllers that would replace driver behavior, carry out the car following task and would satisfy one or more performance and/or safety criteria. Since these studies have essentially been theoretical, they have by necessity required the difficult task of modeling vehicle dynamics. Given a controller for the driver element and a realistic model representation of vehicle dynamics a host of design strategies and issues have been addressed regarding inter-vehicular spacing control, platoon configurations, communication schemes, measurement and timing requirements, protocols, etc. Experimental verifications of these elements are underway at the present time by PATH.

4.6 Summary and Conclusions

Historically, the subject of car following has evolved over the past forty years from conceptual ideas to mathematical model descriptions, analysis, model refinements resulting from empirical testing and evaluation and finally extensions into advanced automatic vehicular control systems. These developments have been overlapping and interactive. There have been ebbs and flows in both the degree of activity and progress made by numerous researchers that have been involved in the contributions made to date.

The overall importance of the development of the subject of car following can be viewed from five vantage points, four of which this chapter has addressed in order. First, it provides a mathematical model of a relative common driving task and provides a scientific foundation for understanding this aspect of the driving task; it provides a means for analysis of local and asymptotic stability in a line of vehicles which carry implications with regard to safety and traffic disruptions and other dynamical characteristics of vehicular traffic flow; it provides a steady state description of single lane traffic flow with associated road capacity estimates; it provides a stepping stone for extension into advance automatic vehicle control systems; and finally, it has and will undoubtedly continue to provide stimulus and encouragement to scientists working in related areas of traffic theory.

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